Individual energy level distributions for one-dimensional diagonal and off-diagonal disorder

2000 J. Phys. A: Math. Gen. 33 6095

(http://iopscience.iop.org/0305-4470/33/35/303)

J. Phys. A: Math. Gen. **33** (2000) 6095–6128. Printed in the UK PII: S0305-4470(00)13436-5

Individual energy level distributions for one-dimensional diagonal and off-diagonal disorder

Christophe Texier

Département de Physique Théorique, Université de Genève, 24 quai Ernest Ansermet, CH-1211 Genève 4, Switzerland

E-mail: texier@kalymnos.unige.ch

Received 19 April 2000

Abstract. We study the distribution of the nth energy level for two different one-dimensional random potentials. This distribution is shown to be related to the distribution of the distance between two consecutive nodes of the wavefunction.

We first consider the case of a white noise potential and study the distributions of energy levels both in the positive and the negative part of the spectrum. It is demonstrated that, in the limit of a large system ($L \to \infty$), the distribution of the nth energy level is given by a scaling law which is shown to be related to the extreme value statistics of a set of independent variables.

In the second part we consider the case of a supersymmetric random Hamiltonian (potential $V(x) = \phi(x)^2 + \phi'(x)$). First we study the case where $\phi(x)$ is a white noise with zero mean. In particular, it is shown that the ground state energy, which behaves on average like $\exp(-L^{1/3})$ in agreement with previous work, is not a self-averaging quantity in the limit $L \to \infty$ as is seen in the case of diagonal disorder. Then we consider the case when $\phi(x)$ has a non-zero mean value.

1. Introduction

One-dimensional disordered systems have been studied in great detail in the past and are still a subject of interest. Examples include recent work on the failure of single-parameter scaling for localization near the band edge or at strong disorder [1], the statistical properties of the Wigner time delay, studied in several works [2–6], as well as the correlations of the time delay for different energies [7]. The distribution of another transport time has also been investigated in [8]. As a last example let us mention that the AC conductivity was re-examined in [9].

In this paper we are interested in the following one-dimensional Hamiltonian:

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{1}$$

where $V(x)$ is a random potential (we choose units such that $\hbar = 2\,m = 1$ for simplicity). The spectral properties of such Hamiltonians were studied in different works for various kinds of disorder. The case of δ-scatterers of random weights and/or at random positions was examined by several authors [10–12] (see also [13]). Here we consider two kinds of random potentials.

(A) In the first case the potential is a white noise, i.e. $V(x)$ is distributed with the Gaussian weight:

$$\mathcal{D}V(x)\; P[V(x)] = \mathcal{D}V(x)\; \exp\left(-\frac{1}{2\sigma}\int \mathrm{d}x\; V(x)^2\right). \tag{2}$$

This implies that $\langle V(x)\rangle = 0$ and $\langle V(x)V(x')\rangle = \sigma\,\delta(x-x')$, all other cumulants being zero. The spectral properties of this model were studied by Halperin [14]; later Berezinskiĭ developed a diagrammatic method adapted to this particular model [15] used to study various quantities in a number of works. We also mention a recent study of the density of states and level statistics of this model using soliton calculus [16]. This model is equivalent to the δ-scatterer model in a certain regime: when the density of impurities is very large compared with their weights. Moreover, it was shown in [17] that results for Gaussian disorder are very generic at high energy: they reproduce what is expected for any random potential provided that the correlation length of this disordered potential (which is zero for the Gaussian disorder) and the de Broglie wavelength are the smallest length scales of the problem, whatever their relative magnitude is.

(B) The second disordered potential we will consider belongs to another class of random potentials, which has attracted the attention of several authors: the case of off-diagonal disorder. The random potential reads

$$V(x) = \phi(x)^2 + \phi'(x) \tag{3}$$

where $\phi(x)$ is random. The Hamiltonian, that which be factorized as $H = Q^\dagger Q$ where $Q = -\mathrm{d}_x + \phi(x)$, describes one-dimensional supersymmetric quantum mechanics [18]. A discrete version of this model would be a one-dimensional (1D) tight-binding Hamiltonian with random hoppings, whereas the discrete version of model (A) is a 1D tight-binding Hamiltonian with random site energies. The interest for the Anderson model with off-diagonal disorder was recently renewed due to its connection with disordered spin chain models [19–21]. It is also worth mentioning the relation of the supersymmetric Hamiltonian with the problem of classical diffusion in random media (see, for example, [22] and references therein); many properties were obtained in this context very recently using a real space renormalization-group method [23]. In the case of off-diagonal disorder, the density of states presents different kinds of behaviour in comparison with diagonal disorder, such as Dyson singularities at the band edge [24] (if $\langle\phi(x)\rangle = 0$), or power-law singularities (if $\langle\phi(x)\rangle \neq 0$) [22, 25–32]. Very recently, the density of states for coupled chains with off-diagonal disorder was studied [33] showing an interesting effect of the parity of the number of chains. For a complete review on one-dimensional disorder systems, the interested reader is referred to [13, 34]. In the following we will consider the case where the random function involved in the potential is distributed according to a Gaussian weight:

$$\mathcal{D}\phi(x)\; P[\phi(x)] = \mathcal{D}\phi(x)\; \exp\left(-\frac{1}{2g}\int \mathrm{d}x\,[\phi(x)-\mu g]^2\right).$$

The purpose of this paper is to study the distribution of the energy E_n of the nth excited state of the Hamiltonian (1) ($(n+1)$th energy level) considered on a finite interval of length L with the wavefunction $\varphi(x)$ satisfying Dirichlet boundary conditions: $\varphi(0) = \varphi(L) = 0$. Let us denote this distribution:

$$W_n(E) = \langle\delta(E - E_n)\rangle \tag{4}$$

where the brackets $\langle\cdots\rangle$ denote an average over different configurations of the disordered potential $V(x)$ (with respect to measure (2) for model (A)). These distributions are related to the average density of states per unit length by the relation

$$\rho(E) = \frac{1}{L}\sum_{n=0}^{\infty} W_n(E). \tag{5}$$

This problem was studied by Grenkova *et al* [35] who derived these distributions for the δ-impurity model of Frisch and Lloyd when the weights of impurities are very large compared with the average density of impurity and the energy. We stress that this is not the limit where this model is equivalent to the model with the white noise potential considered here.

The problem was later addressed by McKean [36] who derived the distribution for the ground state energy ($n = 0$) for $E < 0$; he considered different boundary conditions for the model originally studied by Halperin for a white noise potential. In section 3 we will provide a generalization of the result of McKean by giving the distribution for all eigenvalues in both regions of the spectrum $E < 0$ and $E > 0$, provided $|E| \gg \sigma^{2/3}$. We will demonstrate that the distribution is given by a scaling law in the limit $L \to \infty$ and will show how the parameters scale with the sample size L. This scaling law is similar to the distribution of the extreme value of a set of statistically independent random variables [37]. This is in agreement with the fact that the eigenvalues are not expected to present level repulsion in the limit for which the states are localized, as demonstrated by Molčanov [38].

After having considered case (A) of diagonal disorder we will study in section 4 the problem for off-diagonal disorder (B) at the band edge and at high energy as well. We will consider first the case $\mu = 0$ for which we will study the ground state energy distribution. The analysis for model (A) cannot be applied in this case and we will need to use specific approximations. Note already that our result for the distribution of the ground state energy gives a mean value in agreement with the prediction of Monthus *et al* [39] who showed that the averaged ground state energy behaves like $e^{-L^{1/3}}$ if the system size is very large, by finding a lower bound and an upper bound. Moreover, our result shows that the ground state energy is not a self-averaging quantity as for model (A) at low energy, and its distribution presents a large tail. We also give the distribution $W_n(E)$ in the high-energy limit. Finally, we study the distributions for $\mu \neq 0$ in the low-energy limit.

2. Idea of the method

In this section we give the main ideas of the method we use to derive the eigenvalue distribution. We concentrate on model (A) since the ideas are the same for model (B) apart from small subtleties which we will discuss later.

Let us consider $\psi(x; E)$, the solution of the Schrödinger equation $H\psi(x; E) = E\psi(x; E)$ with the boundary condition $\psi(0; E) = 0$. The boundary condition $\psi(L; E) = 0$ is fulfilled whenever the energy E coincides with an eigenvalue E_n of the Hamiltonian. In this case, the wavefunction $\varphi_n(x) = \psi(x; E_n)/\left[\int_0^L \mathrm{d}x'\, \psi(x'; E_n)^2\right]^{1/2}$ has n nodes in the interval $]0, L[$, and two nodes at the boundaries. Let us denote by ℓ_m the $n+1$ lengths between the nodes. We consider the Ricatti variable

$$z(x; E) = \frac{\mathrm{d}}{\mathrm{d}x} \ln |\psi(x; E)| \tag{6}$$

which obeys the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}x} z = -E - z^2 + V(x) \tag{7}$$

for an initial condition $z(0; E) = +\infty$. This equation may be viewed as a Langevin equation for a particle located at z submitted to a force deriving from the unbounded potential

$$U(z) = Ez + \frac{z^3}{3} \tag{8}$$

and to a random 'force' $V(x)$ (white noise). Each node of the wavefunction corresponds to $|z(x)| = \infty$. At 'time' $x = 0$ the 'particle' starts from $z(0) = +\infty$ and eventually ends at $z(\ell_1 - 0^+) = -\infty$ after a 'time' ℓ_1. Just after the first node it then starts again from $z(\ell_1 + 0^+) = +\infty$, due to the continuity of the wavefunction. It follows from this picture that the distance ℓ_m between two consecutive nodes may be viewed as the 'time' needed by the particle to go through the interval $]-\infty, +\infty[$ (the 'particle' is emitted from $z = +\infty$ at initial 'time' and absorbed when is reaches $z = -\infty$). Following appendix A we introduce the nth moment $\mathcal{L}_n(z)$ of the 'time' $\mathcal{L}$ the particle takes to reach $-\infty$ starting from z:

$$\mathcal{L}_n(z) = \left\langle \mathcal{L}^n \mid z(0) = z;\ z(\mathcal{L}) = -\infty \right\rangle. \tag{9}$$

According to (A9) or (A17) these moments satisfy the following recursion relations:

$$\mathcal{L}_n(z) = \frac{2n}{\sigma} \int_{-\infty}^{z} \mathrm{d}z'\, \mathrm{e}^{\frac{2}{\sigma}U(z')} \int_{z'}^{+\infty} \mathrm{d}z''\, \mathrm{e}^{-\frac{2}{\sigma}U(z'')}\, \mathcal{L}_{n-1}(z'') \tag{10}$$

$$\mathcal{L}_0(z) = 1. \tag{11}$$

The boundary condition at $a = +\infty$ is chosen to be a reflecting one. This choice, which simplifies calculations, is not important since the particle can never go back to this edge of the interval and necessarily ends at $b = -\infty$.

The moments of the lengths ℓ_m between consecutive nodes of the wavefunction are

$$\left\langle \ell^n \right\rangle = \mathcal{L}_n(+\infty). \tag{12}$$

It is worth mentioning that the $n + 1$ lengths are statistically independent because each time the variable z reaches $-\infty$, it loses the memory of its earlier history since it is brought back to the same initial condition and $V(x)$ is δ-correlated. This remark is a crucial point for the derivation of $W_n(E)$.

However, the fact that $V(x)$ is δ-correlated is not essential to ensure the statistical independence of the ℓ_ms. Indeed, imagine that correlations of $V(x)$ are short range, on a scale x_c. Equation (7) shows that when z starts from $+\infty$ at initial 'time', it needs a 'time' Δx to reach the region in z-space where $U(z)$ is of the order of or smaller than σ and where the presence of the random 'force' $V(x)$ matters for the evolution of z. The 'time' Δx, during which the dynamics of z is governed only by the deterministic 'force' $-U'(z)$, can be defined as $\Delta x = \int_{z_0}^{\infty} \frac{\mathrm{d}z}{E+z^2}$ where $U(z_0) = \sigma$; we have $\Delta x \sim 1/k$ for $|E| = k^2 \gg \sigma^{2/3}$, and $\Delta x \sim \sigma^{-1/3}$ for $|E| \ll \sigma^{2/3}$. If z follows the deterministic evolution from $+\infty$ during a 'time' Δx sufficient for $V(x)$ to decorrelate, then the ℓ_ms are statistically independent; this occurs when x_c is much smaller than the smallest length scale among $1/k$ and $\sigma^{-1/3}$.

Let us finally write the equation satisfied by the generating function of the moments of the traversal 'time' $\mathcal{L}$

$$h(\alpha, z) = \left\langle \mathrm{e}^{-\alpha\mathcal{L}} \mid z(0) = z;\ z(\mathcal{L}) = -\infty \right\rangle. \tag{13}$$

According to (A12) it obeys

$$G_z\, h(\alpha, z) = \alpha\, h(\alpha, z) \tag{14}$$

where

$$G_z = -U'(z)\partial_z + \frac{\sigma}{2}\partial_z^2 \tag{15}$$

is the generator of the backward Fokker–Planck equation (BFPE) associated with the stochastic differential equation (7). $-U'(z) = -\partial_z U(z)$ is the force deriving from the potential. The generating function satisfies the boundary conditions:

$$\partial_z h(\alpha, z)|_{z=+\infty} = 0 \tag{16}$$

$$h(\alpha, -\infty) = 1. \tag{17}$$

Coming back to the initial problem, our goal is to compute the probability for the energy of the nth excited state to be E. This occurs if the sum of the $n+1$ distances between the nodes is equal to the length of the system: $L = \sum_{m=1}^{n+1} \ell_m$. As was stated above the ℓ_ms are independent variables and Prob$\left[L = \sum_{m=1}^{n+1} \ell_m\right]$ is given in terms of the distribution of the variables ℓ_m. For the different cases we will analyse throughout this paper, we will initially examine the distribution $P(\ell)$, enabling us to find $W_n(E)$.

3. Diagonal disorder: white noise potential

3.1. Distribution of the distance between consecutive nodes of the wavefunction

We study the distribution $P(\ell)$ of the distance ℓ between two consecutive nodes of the wavefunction. Let us first note that the average integrated density of states per unit length $N(E) = \int_{-\infty}^{E} \mathrm{d}E'\, \rho(E')$ gives the number of states below energy E per unit length, which is also the number of nodes of the wavefunction of energy E per unit length, or in other words the inverse of the average distance between two consecutive nodes:

$$\langle \ell \rangle = \mathcal{L}_1(+\infty) = N(E)^{-1}. \tag{18}$$

The average integrated density of states is given by calculating (10) for $n = 1$ in which it is possible to perform integration over z'' [13, 14]:

$$N(E) = \frac{(\sigma/2)^{1/3}}{\sqrt{\pi}} \left[\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}} \exp\left[-\left(\frac{y^3}{12} + \frac{E}{(\sigma/2)^{2/3}} y \right) \right] \right]^{-1} \tag{19}$$

$$= \frac{(\sigma/2)^{1/3}}{\pi^2} \left[\mathrm{Ai}^2\left(\frac{-E}{(\sigma/2)^{2/3}} \right) + \mathrm{Bi}^2\left(\frac{-E}{(\sigma/2)^{2/3}} \right) \right]^{-1} \tag{20}$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are Airy functions. This result, given by Halperin [14], was first mentioned in [11] as an approximation for the integrated density of states for a potential consisting of randomly dropped δ-scatterers in the limit of high density of scatterers.

We now examine the limit of high energy $|E|/\sigma^{2/3} \to \infty$ in the negative and in the positive part of the spectrum.

3.1.1. Negative part of the spectrum $E = -k^2$: trapping of the Ricatti variable. If $E < 0$ the potential $U(z)$ possesses a local minimum at $z = \sqrt{-E} = k$ that may trap the Ricatti variable (see figure 2). In the limit $|E|/\sigma^{2/3} \gg 1$, the well is very deep (or the diffusion very small) and when travelling from $+\infty$ to $-\infty$, the Ricatti variable spends most of the 'time' in

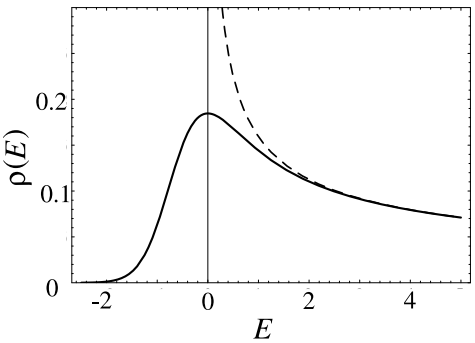

Figure 1. Density of states [14] per unit length for $\sigma = 1$, given by (19). Broken curve, free density of states.

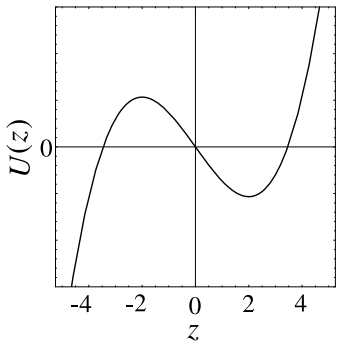

Figure 2. The potential that traps the Ricatti variable z for $\sigma = 1$ and $E = -4$.

the well. Then we expect that the average 'time' $\langle \ell \rangle$ is given by the Arrhenius law and that its distribution is a Poisson distribution, as demonstrated in appendix A. Expanding the potential $U(z)$ in the neighbourhood of its two local extrema

$$U(z) \underset{z\sim k}{\simeq} -\frac{2k^3}{3} + k\,(z-k)^2 \tag{21}$$

$$U(z) \underset{z\sim -k}{\simeq} \frac{2k^3}{3} - k\,(z+k)^2 \tag{22}$$

equation (A22) gives

$$\langle \ell \rangle = \mathcal{L}_1(+\infty) = N(E)^{-1} \simeq \frac{\pi}{\sqrt{-E}} \exp\left[\frac{8}{3\sigma}(-E)^{3/2}\right]. \tag{23}$$

This argument was first used by Jona-Lasinio [40] to find the exponential factor of the integrated density of states. We stress here that it also gives the correct pre-factor, that may be checked by extracting the limiting behaviour of (19).

As we have shown in appendix A the distribution of the time spent in the well is a Poisson law:

$$P(\ell) = N(E) \exp[-\ell\, N(E)]. \tag{24}$$

This equation will be the starting point to find the distribution $W_n(E)$ in section 3.2.

3.1.2. Positive part of the spectrum $E = +k^2$: small-disorder expansion. In the positive part of the spectrum, the potential in which the Ricatti variable evolves has no local extremum (see figure 3) and therefore the length $\langle \ell \rangle$ no longer follows an Arrhenius law.

In the absence of diffusion ($\sigma = 0$) the 'time' needed by the Ricatti variable to go from $+\infty$ to $-\infty$ is $\ell = \pi/k$, which may be found either by integrating (7) in the absence of the potential $V(x)$ or by taking the limit $\sigma \to 0$ in (18) and (19). We can expect that, for sufficiently weak disorder, the length is weakly fluctuating around its mean value. We are now going to show that the distribution of ℓ is a narrow Gaussian distribution in this limit. For this purpose we will analyse the moments $\langle \ell^n \rangle = \mathcal{L}_n(+\infty)$, which is more conveniently achieved by studying a generating function. Instead of the generating function of the moments (13), it is more advantageous to consider the generating function of the cumulants:

$$w(\alpha, z) = \ln h(\alpha, z). \tag{25}$$

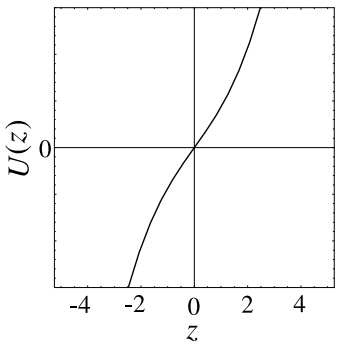

Figure 3. Potential for $\sigma = 1$ and $E = +4$.

It follows from (14) and (15) that

$$-(k^2 + z^2)\partial_z w + \frac{\sigma}{2}\left[\partial_z^2 w + (\partial_z w)^2\right] = \alpha. \tag{26}$$

The solution may be obtained through a perturbative expansion in the parameter σ. We write

$$w = w^{(0)} + w^{(1)} + w^{(2)} + \cdots \tag{27}$$

where $w^{(n)} = \mathrm{O}(\sigma^n)$. The boundary condition for w is: $w(\alpha, z \to -\infty) = 0$. The zeroth order of (26) gives

$$w^{(0)}(\alpha, z) = -\alpha \int_{-\infty}^{z} \mathrm{d}z' \frac{1}{k^2 + z^2} = -\frac{\alpha}{k}\left(\arctan\frac{z}{k} + \frac{\pi}{2}\right). \tag{28}$$

The integral is the 'time' needed to go from z to $-\infty$ in the absence of diffusion as is clear from (7). The nth order of (26) gives

$$w^{(n)}(\alpha, z) = \frac{\sigma}{2}\int_{-\infty}^{z} \frac{\mathrm{d}z'}{k^2 + z'^2}\left[\partial_{z'}^2 w^{(n-1)}(\alpha, z') + \sum_{m=0}^{n-1} \partial_{z'} w^{(m)}(\alpha, z')\, \partial_{z'} w^{(n-1-m)}(\alpha, z')\right]. \tag{29}$$

In particular, we have

$$w^{(1)}(\alpha, z) = \frac{\sigma}{2k^3}\left[-\frac{\alpha}{2k}\frac{1}{(1 + z^2/k^2)^2} + \left(\frac{\alpha}{k}\right)^2 \int_{-\infty}^{z/k} \frac{\mathrm{d}x}{(1 + x^2)^3}\right] \tag{30}$$

$$w^{(2)}(\alpha, z) = \left(\frac{\sigma}{2k^3}\right)^2 \left[\frac{\alpha}{k}\int_{-\infty}^{z/k} \mathrm{d}x \frac{2 - 10x^2}{(1 + x^2)^5} + \left(\frac{\alpha}{k}\right)^2 \frac{5}{4(1 + z^2/k^2)^4} - \left(\frac{\alpha}{k}\right)^3 \int_{-\infty}^{z/k} \frac{2\mathrm{d}x}{(1 + x^2)^5}\right]. \tag{31}$$

Equation (30) gives the dominant contribution to the second cumulant $\langle\langle \ell^2 \rangle\rangle$ of the length. Explicit computation of terms of w of higher order in σ is not required and we only need to know their behaviour with α. Using a recursion argument it is possible to show that $w^{(n)}$ is a polynomial of degree $n + 1$ in α:

$$w^{(n)}(\alpha, z) = \sigma^n \sum_{m=1}^{n+1} \alpha^m\, c_m(z). \tag{32}$$

We now extract from these expressions the information about the cumulants of ℓ:

$$w(\alpha, +\infty) = \sum_{m=1}^{\infty} \frac{(-\alpha)^m}{m!} \langle\langle \ell^m \rangle\rangle. \tag{33}$$

Using (28) and (30) we see that

$$w(\alpha, +\infty) = w^{(0)}(\alpha, +\infty) + w^{(1)}(\alpha, +\infty) + \mathrm{O}(\sigma^2) \tag{34}$$

$$= -\alpha \frac{\pi}{k} + \frac{\alpha^2}{2!} \left(\frac{\pi}{k}\right)^2 \frac{3\sigma}{8\pi k^3} + \mathrm{O}(\sigma^2) \tag{35}$$

then

$$\langle \ell \rangle = \frac{\pi}{k} + \mathrm{O}(\sigma^2) \tag{36}$$

$$\langle\langle \ell^2 \rangle\rangle = \left(\frac{\pi}{k}\right)^2 \frac{3\sigma}{8\pi k^3} + \mathrm{O}(\sigma^2). \tag{37}$$

The nth cumulant is given by the term proportional to α^n in $w(\alpha, \infty)$. Equation (32) shows that the term of lowest order in σ containing α^n is $w^{(n-1)}$, then

$$\langle\langle \ell^n \rangle\rangle = \mathrm{O}(\sigma^{n-1}). \tag{38}$$

The nth-order cumulant for $n > 2$ characterizes fluctuations that are negligible, in the high-energy limit, compared with the fluctuations described by the second cumulant:

$$\frac{\langle\langle \ell^n \rangle\rangle}{\langle\langle \ell^2 \rangle\rangle^{n/2}} = \mathrm{O}\left[\left(\frac{\sigma}{k^3}\right)^{\frac{n}{2}-1}\right] \underset{\sigma/k^3 \to 0}{\longrightarrow} 0. \tag{39}$$

Since the second cumulant is dominating it follows that the distribution of ℓ is Gaussian in this limit

$$P(\ell) = \frac{1}{\sqrt{2\pi \langle\langle \ell^2 \rangle\rangle}} \exp\left[-\frac{(\ell - \langle \ell \rangle)^2}{2\langle\langle \ell^2 \rangle\rangle}\right]. \tag{40}$$

For the positive part of the spectrum, equation (40) shows that the fluctuations of ℓ are small compared with its average value, in contrast to what happens in the negative part of the spectrum where (24) shows that the fluctuations of ℓ are of the same order as the average value.

The structure of the wavefunction. Since we are dealing with one-dimensional disordered system, the wavefunctions are expected to be localized, i.e. decreasing with an exponential damping on a length scale being by definition the localization length λ. Let us recall that $\lambda \simeq \frac{8E}{\sigma}$ for $E \to +\infty$ and $\lambda \simeq \frac{1}{\sqrt{-E}}$ for $E \to -\infty$ (see [17], for example). The results we have derived for the distribution $P(\ell)$ show that in the limit $E \to +\infty$, the consecutive nodes of the wavefunction are separated by weakly fluctuating distances of order π/k, which is much smaller than λ. In contrast, in the limit $E \to -\infty$ the distance ℓ is distributed according to a Poisson law (24) which means that it is probable to find two consecutive nodes as close as possible. However, the typical scale of the length is much larger than the localization length:

$$\ell \sim N(E)^{-1} \simeq \lambda\pi \exp \frac{8}{3\sigma}(-E)^{3/2} \gg \lambda.$$

3.2. Distribution of an individual energy level

We now derive the distribution $W_n(E)$. As we have mentioned above, E coincides with the eigenvalue E_n if the sum of the $n+1$ lengths between nodes is equal to the length L.

3.2.1. Low energy $E = -k^2$. We follow here the idea McKean used to find the ground state energy distribution [36]. The probability that the energy E is between two consecutive energies is

$$\text{Prob}\left[E_{n-1} < E < E_n\right] = \text{Prob}\left[\ell_1 + \cdots + \ell_n < L < \ell_1 + \cdots + \ell_n + \ell_{n+1}\right] \tag{41}$$

$$= \int_0^\infty \mathrm{d}\ell_1\, P(\ell_1) \cdots \int_0^\infty \mathrm{d}\ell_{n+1}\, P(\ell_{n+1})\; \theta\left(\ell_1 + \cdots + \ell_{n+1} - L\right) \times \theta\left(L - \ell_1 - \cdots - \ell_n\right) \tag{42}$$

where $\theta(x)$ is the Heaviside function. Using (24) we obtain

$$\text{Prob}\left[E_{n-1} < E < E_n\right] = \frac{(L\,N(E))^n}{n!}\,\mathrm{e}^{-L\,N(E)}. \tag{43}$$

Introducing the joint distribution for the eigenvalues

$$W_{0,1,\ldots,n,\ldots}(X_0, X_1, \ldots, X_n, \ldots) = \left\langle \prod_{m=0}^{\infty} \delta(X_m - E_m) \right\rangle \tag{44}$$

and differentiating (41) with respect to E gives

$$\frac{\mathrm{d}}{\mathrm{d}E}\,\text{Prob}\left[E_{n-1} < E < E_n\right] = \frac{\mathrm{d}}{\mathrm{d}E} \int \mathrm{d}X_0\, \mathrm{d}X_1 \cdots \mathrm{d}X_n \cdots W_{0,1,\ldots,n,\ldots}(X_0, X_1, \ldots, X_n, \ldots) \times \theta(E - X_{n-1})\; \theta(X_n - E). \tag{45}$$

Differentiation of the integrand of (45) gives two terms. In each term the Heaviside function $\theta(X_n - X_{n-1})$ does not play any role since the joint probability for the different energies is proportional to the following product of Heaviside functions:

$$W_{0,1,\ldots,n,\ldots}(X_0, X_1, \ldots, X_n, \ldots) \propto \theta(X_1 - X_0)\; \theta(X_2 - X_1) \cdots \theta(X_n - X_{n-1}) \cdots. \tag{46}$$

Then we find:

$$\frac{\mathrm{d}}{\mathrm{d}E}\,\text{Prob}\left[E_{n-1} < E < E_n\right] = W_{n-1}(E) - W_n(E). \tag{47}$$

For $n = 0$ we recover the result of McKean [36] for the ground state energy distribution:

$$\int_E^\infty \mathrm{d}E'\, W_0(E') = \mathrm{e}^{-L\,N(E)}. \tag{48}$$

Using (43) and (47), it is now easy to show that:

$$W_n(E) = L\rho(E)\,\frac{(L\,N(E))^n}{n!}\,\mathrm{e}^{-L\,N(E)}. \tag{49}$$

This result has a clear meaning since $L\rho(E)$ gives the probability to find an energy at E, the factor $\frac{x^n}{n!}\mathrm{e}^{-x}$ 'compelling' the number of states below E, $x = LN(E)$, to be close to n.

Let us remark that the relation (5) is satisfied.

We now analyse this result in more detail and demonstrate that the distribution (49) has a scaling form in the limit $L \to \infty$. We use the approximated expression (23) of the integrated density of states per unit length to write the distribution $W_n(E)$ as

$$W_n(E) = \frac{2^{n+3}(n+1)^{n+1}}{n!}\tilde{L}^{n+1}\mathrm{e}^{-g(E)} \tag{50}$$

with

$$g(E) = (n+1)\frac{8(-E)^{3/2}}{3} + 2(n+1)\tilde{L}\sqrt{-E}\,\mathrm{e}^{-\frac{8}{3}(-E)^{3/2}} - \frac{n+2}{2}\ln(-E) \tag{51}$$

where we have introduced

$$\tilde{L} = \frac{L\,\sigma^{1/3}}{2\pi(n+1)}. \tag{52}$$

We set $\sigma = 1$ for simplicity since it will be easy to recover its dependence at the end (the dimension of σ is: $[\sigma] = \text{length}^{-3} = \text{energy}^{3/2}$).

We first look for the typical value of the nth excited state energy E_n^{typ}. It is the solution of the equation $g'(E) = 0$. The variable $Y = 8(-E_n^{\mathrm{typ}})^{3/2}$ is the solution of

$$\tilde{L} = \frac{Y - (n+2)/(n+1)}{Y-1}\,\frac{\mathrm{e}^{Y/3}}{Y^{1/3}}. \tag{53}$$

In the limit $L \to \infty$ we find

$$Y = 3\ln(\tilde{L}) + \ln(3\ln(\tilde{L})) + \mathrm{O}\left[\frac{\ln(3\ln(\tilde{L}))}{\ln(\tilde{L})}\right]$$

that is

$$E_n^{\mathrm{typ}}(L) = -\left(\frac{3\sigma}{8}\ln\tilde{L}\right)^{2/3}\left[1 + \frac{2}{9}\frac{\ln(3\ln\tilde{L})}{\ln\tilde{L}} + \mathrm{O}\left(\frac{\ln^2(3\ln\tilde{L})}{\ln^2\tilde{L}}\right)\right] \tag{54}$$

where we keep the first correction to the dominant term in $\ln^{2/3}\tilde{L}$ because it is larger than the difference between the typical values of two consecutive levels (66). This behaviour was given as a good approximation of E_0 by McKean [36]. Note that the n dependence enters only in $\tilde{L}$.

We will need the maximum value of the distribution, which is

$$W_n(E_n^{\mathrm{typ}}) = \frac{2(n+1)^{n+1}\mathrm{e}^{-(n+1)}}{n!}(3\ln\tilde{L})^{1/3}\left[1 + \mathrm{O}\left(\frac{\ln(3\ln\tilde{L})}{\ln\tilde{L}}\right)\right]. \tag{55}$$

We now study the derivatives of $g(E)$ at $E = E_n^{\mathrm{typ}}$. By definition the first derivative vanishes:

$$g'(E_n^{\mathrm{typ}}) = \mathrm{O}\left(\frac{\ln(3\ln\tilde{L})}{\ln^{2/3}\tilde{L}}\right). \tag{56}$$

After a little of algebra, we find for the mth derivative:

$$g^{(m)}(E_n^{\mathrm{typ}}) = 2^m(n+1)(3\ln\tilde{L})^{m/3}\left[1 + \mathrm{O}\left(\frac{\ln(3\ln\tilde{L})}{\ln\tilde{L}}\right)\right] \qquad m \geqslant 2. \tag{57}$$

The second derivative defines the scale of the fluctuations of E_n:

$$\delta E_n = \frac{1}{\sqrt{g^{(2)}(E_n^{\mathrm{typ}})}} = \sigma^{2/3}\frac{(3\ln\tilde{L})^{-1/3}}{2\sqrt{n+1}}\left[1 + \mathrm{O}\left(\frac{\ln(3\ln\tilde{L})}{\ln\tilde{L}}\right)\right]. \tag{58}$$

Note that the width of $W_n(E)$ decreases as $L \to \infty$. The relative fluctuations tends to zero, that is E_n is self-averaging in this limit.

Moreover, it is possible to introduce the function $\omega_n(X)$:

$$W_n(E) = \frac{1}{\delta E_n}\omega_n\left(\frac{E - E_n^{\mathrm{typ}}}{\delta E_n}\right) \tag{59}$$

with

$$\omega_n(X) = \frac{W_n(E_n^{\mathrm{typ}})}{\sqrt{g^{(2)}(E_n^{\mathrm{typ}})}} \exp\left(-\sum_{m=1}^{\infty}\frac{a_m}{m!}X^m\right) \tag{60}$$

where $a_m = g^{(m)}(E_n^{\mathrm{typ}})/[g^{(2)}(E_n^{\mathrm{typ}})]^{m/2}$. Using (56) and (57) it is easy to see that $a_1 = 0$ and $a_m = (n+1)^{1-m/2}$ in the limit $L \to \infty$. Then we find for the scaling function:

$$\omega_n(X) = \frac{(n+1)^{n+\frac{1}{2}}}{n!} \exp\left(\sqrt{n+1}\ X - (n+1)\,\mathrm{e}^{X/\sqrt{n+1}}\right). \tag{61}$$

A similar result was obtained by Grenkova *et al* [35] for the δ-impurity model of Frisch and Lloyd, in the limit of low impurity density which has no equivalent in the model we consider here. These authors found the same scaling distribution (61) whereas the scaling (54) and (58) is different (it is of course model dependent).

It is interesting to note that (61) is related to extreme value statistics. If we consider a set $\{x_k\}$ of $\mathcal{N} \to \infty$ statistically independent variables distributed according to the same law and order them: $x_0 < x_1 < x_2 < \cdots$, then the distribution of x_n has the form of (61) up to a rescaling. $\omega_0(X)$ is the distribution of the most negative, etc. This problem was studied by Gumbel in 1935. For more details about extreme value statistics, see [37, 41] (see appendix B). The distribution (61) is the extreme value distribution for any distribution of the so-called exponential type (unlimited domain and finite moments). Moreover, the form of the scaling (54) and (58) allows us to find the tail of the distribution $p(E)$ of one of those variables: $p(E) \sim \exp\left[-\frac{8}{3}(-E)^{3/2}\right]$. This tail is not surprising since it is precisely the behaviour of the density of states (see also the discussion in the conclusion). The remark that the eigenvalues are distributed as a set of statistically independent variables is in agreement with the expected absence of level repulsion for one-dimensional disordered systems in the localized regime [35, 38]. Equation (61) allows us to compute the generating function

$$\mathcal{G}_n(k) = \int_{-\infty}^{+\infty} \mathrm{d}X\, \omega_n(X)\, \mathrm{e}^{kX} = \frac{\Gamma(n+1+k\sqrt{n+1})}{\Gamma(n+1)}\ (n+1)^{-k\sqrt{n+1}}. \tag{62}$$

The expansion of the logarithm of the generating function in powers of k gives the cumulants of the distribution (61):

$$\langle X \rangle = \sqrt{n+1}\left[\psi(n+1) - \ln(n+1)\right] \tag{63}$$

$$\langle\langle X^m \rangle\rangle = (n+1)^{m/2}\,\psi^{(m-1)}(n+1) \qquad \text{for} \quad m \geqslant 2 \tag{64}$$

where $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z)$ is the digamma function. It is possible to show that (61) converges to a Gaussian distribution in the large n limit. Using the Stirling formula we find $\ln \mathcal{G}_n(k) \simeq -\frac{k}{2\sqrt{n}} + \frac{k^2}{2}$, i.e.

$$\lim_{n\to\infty} \omega_n(X) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-X^2/2}. \tag{65}$$

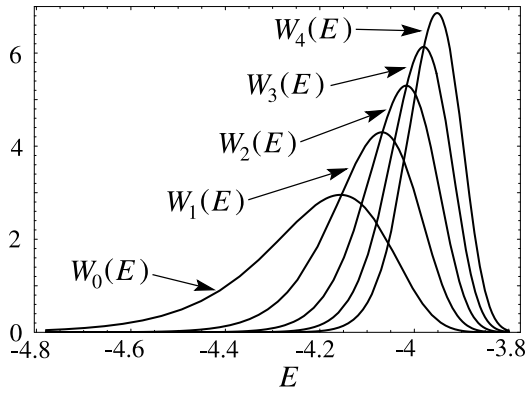

Figure 4. The distributions for the first five energy levels. $\sigma = 1$ and $L = 10^{10}$.

It is also possible to compute the mean level spacing $\langle \Delta_n \rangle$ between the nth excited state and the $(n+1)$th one. Equation (54) shows that:

$$E_{n+1}^{\mathrm{typ}} - E_n^{\mathrm{typ}} = \sigma^{2/3} \frac{1}{2} \ln \frac{n+2}{n+1} \, (3 \ln \tilde{L})^{-1/3} \tag{66}$$

where $\tilde{L}$ is still given by (52) for the nth excited state (and not the length associated with the $(n+1)$th one). Since the mean energy is $\langle E_n \rangle = E_n^{\mathrm{typ}} + \delta E_n \, \langle X \rangle$, we eventually find

$$\langle \Delta_n \rangle = \langle E_{n+1} - E_n \rangle = \frac{\sigma^{2/3}}{2(n+1)} (3 \ln \tilde{L})^{-1/3}. \tag{67}$$

This result shows that the average distance between two consecutive levels is of the same order as the fluctuations (58) of the position of those levels. In other words, the distributions $W_n(E)$ are overlapping functions, as represented in figure 4. Now, if we consider that L is fixed, we can see from (58) and (67) that, apart from the unimportant n dependence of $\tilde{L}$ in the logarithm, the mean level spacing decreases like $\langle \Delta_n \rangle \propto 1/n$, whereas the fluctuations decreases like $\delta E_n \propto 1/\sqrt{n}$ for large n. As n increases for fixed L, the consecutive distributions $W_n(E)$ become more and more overlapped, this is also suggested in figure 4.

3.2.2. High energy $E = +k^2$. In order to find the distribution $W_n(E)$ we use the same idea as before: the probability that $E = E_n$ is related to the probability that the sum of the $n+1$ lengths between the nodes of the wavefunction is L. Let us introduce $\Lambda = \sum_{m=1}^{n+1} \ell_m$. We have $W_n(E)\,\mathrm{d}E = \mathcal{P}_n(\Lambda)\,\mathrm{d}\Lambda$, where $\mathcal{P}_n(\Lambda)$ is the distribution function for Λ. Since the ℓ_ms are statistically independent $\mathcal{P}_n(\Lambda)$ is easily found from (24) or (40) (note that $\mathcal{P}_0(\Lambda) = P(\ell = \Lambda)$). If $E = E_n$, the sum of the lengths between nodes coincides with L: $\Lambda = L$. In this case we have $\Lambda\, N(E) = n+1$. Differentiating this latter equation we get $N(E)\,\mathrm{d}\Lambda + \Lambda\, \rho(E)\,\mathrm{d}E = 0$, that is $\frac{\mathrm{d}\Lambda}{\mathrm{d}E} = -\frac{\Lambda\, \rho(E)}{N(E)}$. It follows that

$$W_n(E) = \frac{L\, \rho(E)}{N(E)} \, \mathcal{P}_n(\Lambda = L). \tag{68}$$

Using (24), this equation allows us to get straightforwardly (49). In the positive part of the spectrum, since the distribution (40) is Gaussian, the distribution $\mathcal{P}_n(\Lambda)$ is also Gaussian and

we obtain

$$W_n(E) = \frac{L\,\rho(E)}{\sqrt{2\pi(n+1)N(E)^2\langle\langle \ell^2\rangle\rangle}} \exp\left[-\frac{(L\,N(E)-n-1)^2}{2N(E)^2\langle\langle \ell^2\rangle\rangle}\right] \tag{69}$$

with $\langle\langle \ell^2\rangle\rangle$ being given by (37).

We now go to the variable $k = \sqrt{E}$ for clarity. We write

$$W_n(E)\,\mathrm{d}E = \mathrm{d}k\,\frac{1}{\sqrt{2\pi s(k)}} \exp\left[-\frac{(k-(n+1)\frac{\pi}{L})^2}{2s(k)}\right] \tag{70}$$

with

$$s(k) = (n+1)\frac{3\pi}{8}\frac{\sigma}{k^3L^2}.$$

The maximum value of the distribution corresponds to $k = k_n^0 = (n+1)\frac{\pi}{L}$, which is the de Broglie wavelength in the absence of the disordered potential. We remember that we are dealing with a high-energy limit $E \sim \frac{(n+1)^2}{L^2} \gg \sigma^{2/3}$, which may be conveniently written as

$$\frac{1}{n+1}\frac{L}{\lambda_n} \ll 1 \tag{71}$$

where we have introduced the localization length λ_n associated with an energy $E_n^0 = (k_n^0)^2 = (n+1)^2\frac{\pi^2}{L^2}$. We recall that $\lambda = \frac{8E}{\sigma}$ for $E \gg \sigma^{2/3}$ [17]. The condition (71) is fulfilled either for a delocalized regime $L \ll \lambda$, or for a localized regime $L \gg \lambda$ provided that n is sufficiently large. Note that there is no restriction on L for the derivation of $W_n(E)$.

Due to the condition of validity (71), it is possible to neglect the dependence of $s(k)$ on k and replace $s(k)$ by $s(k_0^n)$; then the distribution $W_n(E)$ is a Gaussian distribution

$$W_n(E) = \frac{1}{\sqrt{2\pi\delta E_n^2}} \exp\left[-\frac{(E-\langle E_n\rangle)^2}{2\delta E_n^2}\right] \tag{72}$$

of mean

$$\langle E_n\rangle = (n+1)^2\frac{\pi^2}{L^2} \tag{73}$$

and width

$$\delta E_n = \sqrt{\frac{3\sigma}{2L}}. \tag{74}$$

According to (71) the relative fluctuations are necessarily small:

$$\frac{\delta E_n^2}{\langle E_n\rangle^2} = \frac{3}{\pi^2(n+1)^2}\frac{L}{\lambda_n} \ll 1.$$

More interesting is to compare these fluctuations to the mean level spacing

$$\langle\Delta_n\rangle = \langle E_{n+1} - E_n\rangle \simeq 2(n+1)\frac{\pi^2}{L^2}.$$

Then

$$\frac{\delta E_n^2}{\langle\Delta_n\rangle^2} \simeq \frac{3}{\pi^2}\frac{L}{\lambda_n}. \tag{75}$$

This condition tells us that the fluctuations δE_n of the position of E_n are larger than the mean level spacing $\langle \Delta_n \rangle$ if we are in a localized regime; the distributions $W_n(E)$ are overlapping functions. This agrees with the fact that no level repulsion is expected in this regime [35, 38] where the level spacing is believed to be distributed according to a Poisson law [38]. In the delocalized regime $L \ll \lambda$, the fact that the distributions $W_n(E)$ are non-overlapping indicate level repulsion.

It is worth mentioning that (73) and (74) may be found by a simpler, although less systematic, perturbative argument: the energy of the $(n+1)$th level is, up to first-order perturbation theory $E_n \simeq E_n^0 + \langle \psi_n^0 | V(x) | \psi_n^0 \rangle$, with $\psi_n^0(x) = \sqrt{\frac{2}{L}} \sin k_n^0 x$. Then it is straightforward to see that $\langle E_n \rangle$ is given by (73) and that

$$\delta E_n^2 = \left\langle \left[\int_0^L \mathrm{d}x \; V(x) \psi_n^0(x)^2 \right]^2 \right\rangle = \sigma \int_0^L \mathrm{d}x \; \psi_n^0(x)^4$$

leads to (74).

Let us end the section with a remark concerning the ground state energy. We have noted that the ground state energy is a self-averaging quantity ($\langle E_0 \rangle \gg \delta E_0$). Its behaviour with L is $E_0 \sim -\ln^{2/3} L$ for $L \to \infty$, then it is vanishing $E_0 \sim 0$ for a size $L \sim \sigma^{-1/3}$ and behaving like $E_0 \sim 1/L^2$ for $L \to 0$.

4. Off-diagonal disorder: supersymmetric random Hamiltonian

The Hamiltonian we consider in this section is the following supersymmetric Hamiltonian:

$$H_S = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 + \phi'(x). \tag{76}$$

The spectral and localization properties of this Hamiltonian were studied for various kinds of disorder. The case of white noise was analysed in [22, 25, 26]. The case of a random telegraph process was also studied in detail [27, 28]. Note that the Hamiltonian (76) is the square of a Dirac Hamiltonian with a random mass. The spectral properties of 1D random Dirac Hamiltonians were studied in a more general situation by Bocquet [31, 32]. His analysis was extended very recently to study the distribution function of the local density of states [42]; these authors used replica trick and supersymmetry and obtained for the supersymmetric Hamiltonian in a high-energy limit a result similar to that derived by Altshuler and Prigodin [43] for model (A) using the Berezinskiĭ technique.

As was mentioned in the introduction, we focus on the situation where $\phi(x)$ is a white noise:

$$\mathcal{D}\phi(x) \; P[\phi(x)] = \mathcal{D}\phi(x) \; \exp\left[-\frac{1}{2g} \int \mathrm{d}x \; [\phi(x) - \mu g]^2 \right]. \tag{77}$$

We recall that the integrated density of states is in this case [13, 22, 25, 26]:

$$N(E) = \frac{2g}{\pi^2} \frac{1}{J_\mu^2(\sqrt{E}/g) + N_\mu^2(\sqrt{E}/g)} \tag{78}$$

where $J_\nu(z)$ and $N_\nu(z)$ are the Bessel functions of first and second kind, respectively.

We now consider the situation where $\mu = 0$ and will discuss the case $\mu \neq 0$ in the last section. In general, the Hamiltonian (76) may possess a zero mode, however, for the problem of interest here, it cannot satisfy the Dirichlet boundary conditions; we say that the supersymmetry is broken. Due to the supersymmetric structure of the Hamiltonian the spectrum of H_S is

positive. It is possible to rewrite the Schrödinger equation $H_S\varphi(x) = k^2\varphi(x)$ as two coupled first-order differential equations:

$$Q^\dagger \chi(x) = k\varphi(x) \tag{79}$$

$$Q\varphi(x) = k\chi(x) \tag{80}$$

where $Q = -\mathrm{d}_x + \phi(x)$ and $Q^\dagger = \mathrm{d}_x + \phi(x)$. As in the previous section, the first step is to study the statistics of the distances between the nodes of the wavefunction. For the supersymmetric Hamiltonian it is not very convenient to consider the 'Ricatti' variable $z = \frac{\chi}{\varphi}$ because it is not an additive process but a multiplicative process: $\mathrm{d}_x z = k + kz^2 - 2\phi(x)z$. To help the discussion we introduce two intermediate variables, a phase variable $\vartheta(x)$ and an envelope function $\exp\xi(x)$:

$$\varphi(x) = \mathrm{e}^{\xi(x)} \sin\vartheta(x) \tag{81}$$

$$\chi(x) = -\mathrm{e}^{\xi(x)} \cos\vartheta(x). \tag{82}$$

These two functions obey the set of coupled stochastic differential equations, written in the Stratonovich convention:

$$\frac{\mathrm{d}}{\mathrm{d}x}\vartheta = k + \phi(x)\sin 2\vartheta \qquad \text{(Stratonovich)} \tag{83}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\xi = -\phi(x)\cos 2\vartheta \qquad \text{(Stratonovich).} \tag{84}$$

The fact that the periodicity in ϑ of these two equations is $\pi/2$, and that $\phi(x)$ is δ-correlated, means that the distance separating the two points where $\vartheta = m\pi/2$ and $(m+1)\pi/2$ depends only on $\phi(x)$ between these two points (the periodicity in ϑ is $\pi/2$ and not π because the sign can always be absorbed in the function $\phi(x)$ if it is a white noise of zero mean). Then the statistically independent random variables to be considered in a first step are not the distances ℓ between successive nodes but rather the distances, denoted by Λ, separating the points where $\vartheta(x)$ takes a value equal to a multiple integer of $\pi/2$ (the point where $(\vartheta(x) \bmod \pi) = \pi/2$ corresponds to a local extremum of the oscillatory part of the wavefunction). The length ℓ between two consecutive nodes of the wavefunction is then the sum of two independent Λs.

It is more convenient to introduce a process which is additive in the noise $\phi(x)$. It is easy to see that this is achieved by the change of variable $\zeta(x) = \frac{1}{2}\ln|\tan\vartheta(x)|$. This variable obeys the stochastic differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\zeta = k\cosh 2\zeta + \phi(x). \tag{85}$$

This is a Langevin equation for a 'particle' of position ζ travelling from $-\infty$ to $+\infty$ in a potential

$$U(\zeta) = -\frac{k}{2}\sinh 2\zeta \tag{86}$$

(see figure 5) and feeling the random 'force' $\phi(x)$. Λ is the 'time' the 'particle' needs to go through the interval. As for model (A), we are going to study the statistical properties of the distances ℓ between the nodes, sum of two independent variables Λ. The knowledge of the distribution of ℓ will allow us to find the distribution of the energies.

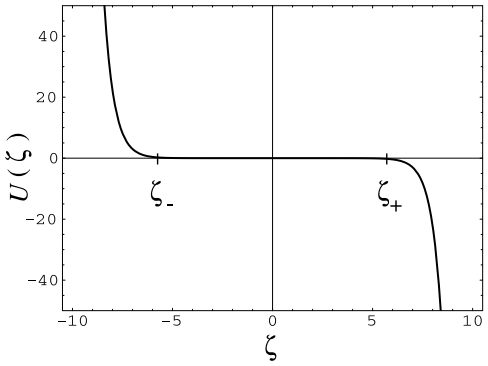

Figure 5. Potential 'seen' by the variable ζ for $k = 10^{-5}$.

4.1. Distribution of distances between the nodes

We introduce the moments of the 'time' needed by the variable $\zeta(x)$ to reach $+\infty$ starting from the position ζ:

$$\tilde{\Lambda}_n(\zeta) = \langle \tilde{\Lambda}^n \mid \zeta(0) = \zeta;\ \zeta(\tilde{\Lambda}) = +\infty \rangle. \tag{87}$$

They are given by (see appendix A):

$$\tilde{\Lambda}_n(\zeta) = \frac{2n}{g} \int_{\zeta}^{+\infty} \mathrm{d}\zeta'\, \mathrm{e}^{\frac{2}{g}U(\zeta')} \int_{-\infty}^{\zeta'} \mathrm{d}\zeta''\, \mathrm{e}^{-\frac{2}{g}U(\zeta'')}\, \tilde{\Lambda}_{n-1}(\zeta'') \tag{88}$$

$$\tilde{\Lambda}_0(\zeta) = 1. \tag{89}$$

The moments of the random variable of interest are $\langle \Lambda^n \rangle = \tilde{\Lambda}_n(-\infty)$.

4.1.1. Low-energy limit: $E \ll g^2$. At low energy $k \ll g$, the potential is still a monotonic function and there is no process of trapping of the variable ζ by a well as for the low-energy limit of model (A), which means that we have to develop a specific approximation scheme. For this purpose we start by studying the average time $\tilde{\Lambda}_1(\zeta)$ to go from ζ to $+\infty$:

$$\tilde{\Lambda}_1(\zeta) = \frac{2}{g} \int_{\zeta}^{+\infty} \mathrm{d}\zeta'\, \exp\left[-\frac{k}{g}\sinh 2\zeta'\right] \int_{-\infty}^{\zeta'} \mathrm{d}\zeta''\, \exp\left[\frac{k}{g}\sinh 2\zeta''\right]. \tag{90}$$

We introduce $\zeta_{\pm}$, the two solutions of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} \exp\left(\frac{k}{g}\sinh 2\zeta\right) = 0.$$

In the low-energy limit $\zeta_{\pm} \simeq \pm\frac{1}{2}\ln\frac{g}{k}$. The positions $\zeta_{\pm}$ are the crossover points where the force deriving from the potential $U(\zeta)$ is of the same order as the random 'force' $\phi(x)$. The study of (90) leads us to distinguish three regions to which the initial condition can belong:

(a) $\zeta_+ < \zeta$

$$\tilde{\Lambda}_1(\zeta) \simeq \frac{1}{g}\mathrm{e}^{-2(\zeta-\zeta_+)} \tag{91}$$

(b) $\zeta_- < \zeta < \zeta_+$

$$\tilde{\Lambda}_1(\zeta) \simeq \frac{1}{g}\left[(\zeta_+ - \zeta_-)^2 - (\zeta - \zeta_-)^2\right] + \frac{1}{g} \tag{92}$$

(c) $\zeta < \zeta_-$

$$\tilde{\Lambda}_1(\zeta) \simeq \frac{1}{g}\left[1 - \mathrm{e}^{2(\zeta-\zeta_-)}\right] + \frac{1}{g}(\zeta_+ - \zeta_-)^2 + \frac{1}{g}. \tag{93}$$

What can we learn from these behaviours? (a) If ζ starts from the neighbourhood of ζ_+, it needs a 'time' of order $1/g$ to reach $+\infty$. (b) When ζ starts somewhere in $[\zeta_-, \zeta_+]$ the 'time' needed to reach $+\infty$ is dominated by the first term, which behaves like ζ^2, i.e. like the 'time' for a free diffusive 'particle'. (c) When ζ starts from $-\infty$ it needs a 'time' $1/g$ to reach ζ_-; then ζ travels from ζ_- to ζ_+ in a 'time' $\frac{(\text{distance})^2}{\text{diffusion}} = \frac{1}{g}(\zeta_+ - \zeta_-)^2 \simeq \frac{1}{g}\ln^2\frac{g}{k}$ and eventually ends at $+\infty$ after an additional 'time' $1/g$.

The physical picture for the motion of the fictitious particle of position ζ we get from these results is now very clear. ζ travels from $-\infty$ to ζ_- very quickly due to the potential only; for $\zeta \in [\zeta_-, \zeta_+]$ the potential becomes negligible compared with the random force which is of order g and ζ evolves due to the random force only. It increases like the position of a free diffusive particle until it reaches ζ_+ from where it goes very rapidly to $+\infty$.

We are now going to use this picture to study the distribution of Λ. We are interested in the characteristic function for the travelling time

$$h(\alpha, \zeta) = \langle \mathrm{e}^{-\alpha\tilde{\Lambda}} \mid \zeta(0) = \zeta;\ \zeta(\tilde{\Lambda}) = +\infty \rangle. \tag{94}$$

From the Langevin equation (85) we get the BFPE generator and we see that h obeys (see appendix A)

$$\left(k\cosh 2\zeta\ \partial_\zeta + \frac{g}{2}\partial_\zeta^2\right) h(\alpha, \zeta) = \alpha\ h(\alpha, \zeta) \tag{95}$$

with boundary conditions:

$$\partial_\zeta h(\alpha, \zeta)\big|_{-\infty} = 0 \tag{96}$$

$$h(\alpha, +\infty) = 1. \tag{97}$$

Since the travelling 'time' is dominated by the 'time' spent in the region $[\zeta_-, \zeta_+]$ where the diffusion is free, it means that the characteristic function in the limit $k \ll g$ is approximatively given by the solution of the equation for the free diffusion on this finite interval. Then equations (95)–(97) may be replaced by the following equation:

$$\frac{g}{2}\partial_\zeta^2 h(\alpha, \zeta) = \alpha\ h(\alpha, \zeta) \tag{98}$$

with boundary conditions:

$$\partial_\zeta h(\alpha, \zeta_-) = 0 \tag{99}$$

$$h(\alpha, \zeta_+) = 1. \tag{100}$$

The first condition is a reflection condition since the variable ζ 'sees' a steep wall in ζ_-. The second condition is an absorption condition since as ζ reaches ζ_+, it eventually ends to $+\infty$ after a negligible time. Since we are dealing with free diffusion, the solution is then very easy to find:

$$h(\alpha, \zeta) = \frac{\cosh\left[\sqrt{2\alpha/g}(\zeta - \zeta_-)\right]}{\cosh\left[\sqrt{2\alpha/g}(\zeta_+ - \zeta_-)\right]}. \tag{101}$$

The characteristic function for the length Λ is

$$\langle \mathrm{e}^{-\alpha\Lambda} \rangle = h(\alpha, \zeta_-) = \frac{1}{\cosh\sqrt{\alpha B}} \tag{102}$$

where we have introduced $B = \frac{2}{g}\ln^2\frac{g}{k} = \frac{1}{2g}\ln^2\frac{g^2}{E}$. Since we are, in fact, interested in the distance ℓ between the nodes of the wavefunction, which is given as the sum of two independent Λs, we give the characteristic function for ℓ:

$$\langle \mathrm{e}^{-\alpha\ell}\rangle = \frac{1}{\cosh^2\sqrt{\alpha B}}. \tag{103}$$

The inverse Laplace transform gives the distribution function:

$$P(\ell) = \frac{1}{B}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\frac{\mathrm{d}q}{2\mathrm{i}\pi}\frac{\mathrm{e}^{q\ell/B}}{\cosh^2\sqrt{q}} \tag{104}$$

where the integral is taken over a Bromwitch contour. $\cosh\sqrt{q}$ is an analytic function in the variable q, whose zeros are: $q_m = -\frac{\pi^2}{4}(2m+1)^2$ for $m \in \mathbb{N}$ (note that there is no branch cut since cosh is an even function). We can expand $\cosh\sqrt{q}$ in the neighbourhood of its zeros and we obtain

$$\cosh\sqrt{q} \underset{q\sim q_m}{\simeq} \frac{(-1)^m}{\pi(2m+1)}(q-q_m)\left[1+\frac{1}{\pi^2(2m+1)^2}(q-q_m)+\cdots\right]. \tag{105}$$

We use the residue theorem to evaluate integral (104); we can indeed check that the contribution of integrals over the large semi-circles needed to close the contour in the complex plane are vanishing in the limit of their radius going to infinity. Then we obtain

$$P(\ell) = \theta(\ell)\frac{1}{B}\sum_{m=0}^{\infty}\left[\frac{\ell}{B}\pi^2(2m+1)^2-2\right]\mathrm{e}^{-\frac{\pi^2}{4}(2m+1)^2\ell/B}. \tag{106}$$

Using the identities [44]:

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$$

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4} = \frac{\pi^4}{96}$$

and

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^6} = \frac{\pi^6}{960}$$

we can check the normalization and get the two first cumulants:

$$\langle\ell\rangle = B = \frac{1}{2g}\ln^2\frac{g^2}{E} \tag{107}$$

$$\langle\langle\ell^2\rangle\rangle = \frac{1}{3}\langle\ell\rangle^2. \tag{108}$$

The average distance between consecutive nodes is the inverse of the integrated density of states. We can check from the exact result (78) that the limit behaviour of $N(E)$ is indeed in perfect agreement with the result we find here $\langle\ell\rangle = N(E)^{-1} \simeq \frac{1}{2g}\ln^2\frac{g^2}{E}$, which gives us a certain confidence in the approximation we have made to calculate the characteristic function $h(\alpha,\zeta)$. Since $B = \langle\ell\rangle$ it can be replaced by $N(E)^{-1}$ and we can rewrite the distribution as

$$P(\ell) = N(E)\,\varpi_0(N(E)\ell) \tag{109}$$

where

$$\varpi_0(X) = \theta(X) \sum_{m=0}^{\infty} \left[\pi^2(2m+1)^2 X - 2\right] \mathrm{e}^{-\frac{\pi^2}{4}(2m+1)^2 X} \tag{110}$$

replaces the exponential function obtained for the low-energy limit in model (A) (24).

We can extract the limiting behaviours of the distribution (106). For this purpose we introduce the θ-function (not to be confused with the Heaviside function $\theta(x)$)

$$\tilde{\theta}(x) = \sum_{m=0}^{\infty} \mathrm{e}^{-(2m+1)^2 x} = \frac{1}{4}\sqrt{\frac{\pi}{x}} \left(1 + 2\sum_{m=1}^{\infty} (-1)^m \mathrm{e}^{-m^2 \frac{\pi^2}{4x}}\right) \tag{111}$$

(this is the elliptic theta function $\vartheta_1(\pi/2 \,|\, 4\mathrm{i}x/\pi)$ [44, 8.180]). The distribution (110) is related to the θ-function by

$$\varpi_0(X) = -2\left(2X\frac{\partial}{\partial X} + 1\right)\tilde{\theta}\left(\frac{\pi^2}{4}X\right). \tag{112}$$

Using (111) and (112) we can find an expression adapted for the limit $X \to 0$:

$$\varpi_0(X) = \frac{4}{\sqrt{\pi}} \frac{\theta(X)}{X^{3/2}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2 \mathrm{e}^{-m^2/X} \underset{X\to 0}{\simeq} \frac{4}{\sqrt{\pi}} \frac{\theta(X)}{X^{3/2}} \mathrm{e}^{-1/X}. \tag{113}$$

The tail of the distribution is exponential and dominated by the term $m = 0$ in (110):

$$\varpi_0(X) \underset{X\to\infty}{\simeq} \pi^2 X \mathrm{e}^{-\frac{\pi^2}{4}X}. \tag{114}$$

Structure of the wavefunction at low energy. It is well known that the supersymmetric Hamiltonian (76) exhibits a delocalization transition at zero energy [22]. As energy goes to zero, the localization length behaves like $\lambda \simeq \frac{1}{g}\ln(4g^2/E)$. The distribution we have just found for the distances between the nodes shows that two consecutive nodes are separated by a distance of order $B = \frac{1}{g}\ln^2(g^2/E)$ which is much larger than the localization length λ. It is also worth mentioning that this distance is the correlation length appearing in the average Green function [22]. In contrast to what happens for model (A) at low energy, where the nodes of the wavefunction can be arbitrarily close, the nodes of the wavefunction for the supersymmetric model (B) with $\mu = 0$ are extremely unlikely to be closer than a distance of order $\frac{1}{g}\ln^2(g^2/E)$: the behaviour of the distribution $P(\ell)$ (figure 6) indicates a 'repulsion' of the nodes of the wavefunction.

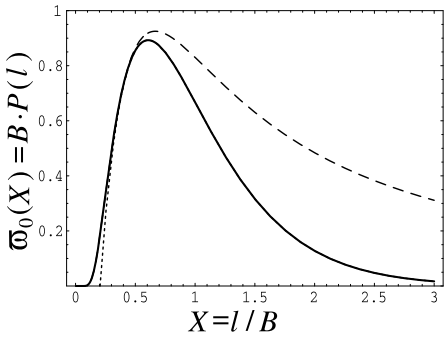

Figure 6. Function $\varpi_0(X)$ giving the distribution of the distance between consecutive nodes of the wavefunction for the supersymmetric Hamiltonian in the limit $E \to 0$. Broken curve, term $m = 1$ of (113); Dotted curve, term $m = 0$ of (110).

4.1.2. High-energy limit ($E \gg g^2$): small-disorder expansion. At high energy we perform the same perturbative analysis in the disordered strength g as for model (A), hence we do not enter into the details. The generating function for the cumulants $w(\alpha, \zeta) = \ln h(\alpha, \zeta)$ obeys

$$k \cosh 2\zeta \; \partial_\zeta w + \frac{g}{2}\left[\partial_\zeta^2 w + (\partial_\zeta w)^2\right] = \alpha. \tag{115}$$

We solve this equation perturbatively in g: $w = w^{(0)} + w^{(1)} + \cdots$, where $w^{(n)} = \mathrm{O}(g^n)$. Since the reflection condition is at $\zeta = -\infty$ and the absorbing one at $\zeta = +\infty$, we now have $w(\alpha, +\infty) = 0$ to ensure $h(\alpha, +\infty) = 1$. To zeroth order we find

$$w^{(0)}(\alpha, \zeta) = -\frac{\alpha}{k}\int_\zeta^{+\infty} \mathrm{d}\zeta' \, \frac{1}{\cosh 2\zeta'} \tag{116}$$

and to order n:

$$w^{(n)}(\alpha, \zeta) = \frac{g}{2k}\int_\zeta^{\infty} \frac{\mathrm{d}\zeta'}{\cosh 2\zeta'}\left[\partial_{\zeta'}^2 w^{(n-1)}(\alpha, \zeta') + \sum_{m=0}^{n-1} \partial_{\zeta'} w^{(m)}(\alpha, \zeta') \, \partial_{\zeta'} w^{(n-1-m)}(\alpha, \zeta')\right]. \tag{117}$$

The information of interest is contained in the first two orders:

$$w(\alpha, -\infty) = -\alpha\frac{\pi}{2k} + \frac{\alpha^2}{2!}\frac{\pi g}{4k^3} + \mathrm{O}(g^2) \tag{118}$$

that give the cumulants of the variable Λ. Let us recall that the length ℓ is the sum of two independent Λs, then:

$$\langle \ell \rangle = \frac{\pi}{k} + \mathrm{O}(g^2) \tag{119}$$

$$\langle\langle \ell^2 \rangle\rangle = \left(\frac{\pi}{k}\right)^2 \frac{g}{2\pi k} + \mathrm{O}(g^2). \tag{120}$$

Since the cumulants of higher orders are small, $\langle\langle \ell^n \rangle\rangle = \mathrm{O}(g^{n-1})$, the distribution of ℓ is Gaussian in the small-disorder limit as for the high-energy limit of model (A), given by (40).

4.2. Distribution of energy level

4.2.1. Low energy: $E \ll g^2$. We first consider the distribution for the ground state energy and will give an explicit form for the excited state energy distribution without going further into the calculation to avoid increasing technical complexity. The probability for the ground state to be at a given energy is proportional to the probability that the distance between the two first nodes of the solution of the Schrödinger equation is equal to the length of the disordered region. According to (68) we have

$$W_0(E) = \frac{L\rho(E)}{N(E)} P(\ell = L). \tag{121}$$

Using (109), we see that this distribution is a scaling function of the variable:

$$X = L\, N(E) = \frac{2gL}{\ln^2(g^2/E)} \tag{122}$$

which is the averaged number of states below E for a system of length L. Then

$$W_0(E) = L\rho(E)\; \varpi_0(L\, N(E)). \tag{123}$$

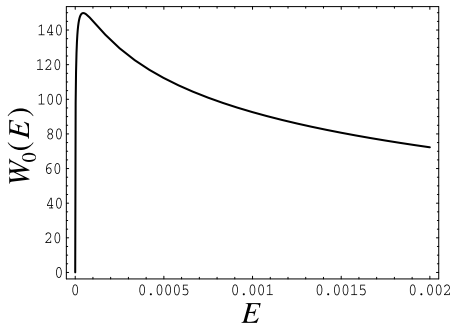

Figure 7. Distribution $W_0(E)$ for the supersymmetric potential with $\mu = 0$. $L = 10$, $g = 1$.

From (113) and (114) we see that the distribution $W_0(E)$ presents a log-normal behaviour at low energy

$$W_0(E) \simeq \frac{8}{\sqrt{2\pi gL}} \frac{1}{E} \exp\left[-\frac{\ln^2(g^2/E)}{2gL}\right] \qquad \text{for} \quad E \ll g^2 \mathrm{e}^{-\sqrt{2gL}} \tag{124}$$

and the following behaviour at large E (however, smaller than g^2, not to be out of the range of validity of the approximation we have made):

$$W_0(E) \simeq \frac{8\pi^2 g^2 L^2}{E \ln^5(g^2/E)} \exp\left[-\frac{\pi^2 gL}{2\ln^2(g^2/E)}\right] \qquad \text{for} \quad g^2 \mathrm{e}^{-\sqrt{2gL}} \ll E \ll g^2 \tag{125}$$

(see figure 7).

We may now proceed to study several aspects of the distribution $W_0(E)$. The typical value of the distribution is given by the limiting behaviour of the distribution for small E (124):

$$E_0^{\mathrm{typ}} \simeq g^2 \mathrm{e}^{-gL}. \tag{126}$$

It is also interesting to estimate the median value: $\int_0^{E_0^{\mathrm{med}}} \mathrm{d}E\ W_0(E) = \frac{1}{2}$. Assuming that we can also consider the limiting behaviour (124), we find that it is the solution of

$$\Phi\left(\frac{\ln(g^2/E_0^{\mathrm{med}})}{\sqrt{2gL}}\right) \simeq \tfrac{7}{8}$$

where $\Phi(z)$ is the error function. Then

$$E_0^{\mathrm{med}} \sim g^2 \mathrm{e}^{-c\sqrt{gL}} \tag{127}$$

(c is a numerical factor) which is at the boundary of the domain where approximation (124) holds.

We may also compute the moments of the ground state energy:

$$\langle E_0^\beta \rangle = \int_0^\infty \mathrm{d}E\ E^\beta\ W_0(E) = \int_0^\infty \mathrm{d}X\ \varpi_0(X)\, \mathrm{e}^{-\beta\sqrt{2gL/X}}. \tag{128}$$

If $\beta > 0$ the exponential select the tail (114) of the distribution

$$\langle E_0^\beta \rangle \simeq \pi^2 \int_0^\infty \mathrm{d}X\ X\ \exp\left[-\frac{\pi^2}{4}X - \beta\sqrt{\frac{2gL}{X}}\right]. \tag{129}$$

This integral is easily worked out by the steepest-descent method, and we eventually find:

$$\langle E_0^\beta \rangle \underset{L\to\infty}{\simeq} \frac{16\beta g^{2\beta}}{\sqrt{6\pi}} (gL)^{1/2} \exp\left[-\frac{3}{2}(\pi^2\beta^2 gL)^{1/3}\right] \qquad \text{for} \quad \beta > 0. \tag{130}$$

In particular, the mean value of the ground state energy behaves like $\langle E_0 \rangle \sim \mathrm{e}^{-\tilde{c}\,L^{1/3}}$, which is much larger than the median value. This result is in agreement with the work of Monthus *et al* [39] who found upper and lower bounds using a perturbative expression for the ground state energy as a functional of $\phi(x)$. The atypical behaviour of the positive moments with n like $\langle {E_0}^n \rangle \sim b^{n^{2/3}}$ may also be recovered with the method used in [39] as remarked by Oshanin [45].

It is also possible to compute the negative moments. If $\beta < 0$, the exponential in (128) selects the origin of the distribution. The steepest-descent method gives

$$\langle E_0^\beta \rangle \underset{L\to\infty}{\simeq} 8g^{2\beta} \exp\left(\frac{1}{2}\beta^2 g L\right) \qquad \text{for} \quad \beta < 0 \tag{131}$$

which presents a dependence in β characteristic of the log-normal behaviour (124).

We now make some remarks.

(a) It is worth mentioning that the fluctuations behave at large L like $\delta E_0^2 \sim \exp\left[-\frac{3}{2}(4\pi^2 g L)^{1/3}\right]$ and are much larger than the mean value $\langle E_0 \rangle^2 \sim \exp\left[-\frac{3}{2}(8\pi^2 g L)^{1/3}\right]$. Thus the ground state energy is not a self-averaging quantity in the $L \to \infty$ limit as was the case for model (A).

(b) A remark related to the previous one. Since the distribution $W_0(E)$ is a scaling function of $L/\ln^2(1/E)$, it becomes more and more peaked near zero when $L \to \infty$. Moreover, it is rather obvious that it cannot be written as a scaling function of $(E - E_1(L))/E_2(L)$ where $E_{1,2}(L)$ would take into account the L dependence as was possible for model (A). In other words, none of the moments (130) determines the nature of the fluctuations at $L \to \infty$ since $\langle E_0^\beta \rangle^{1/\beta} \sim L^{1/2\beta} \exp\left[-\frac{3}{2}(\pi^2 L/\beta)^{1/3}\right]$ has a non-trivial dependence in β.

(c) It is interesting to note that the average Green function, derived in [22], presents the same kind of behaviour as the distribution $W_0(E)$:

$$\left\langle \langle x \,| \frac{1}{E - H_S + \mathrm{i}0^+} | \, x' \rangle \right\rangle \simeq \sum_m c_m \exp\left(-\frac{\pi^2}{2\ln^2 E}(2m+1)^2 |x - x'|\right)$$

(in the limit $E \to 0$ with $g = 1$).

(d) The distribution (110) was obtained in [23] in the context of classical diffusion by using a real space renormalization-group method. In this case the distribution is interpreted as the distribution of the smallest relaxation time.

(e) The distribution (110) has still another interpretation: it is related to the distribution of the span of a Brownian motion (see [39] and references therein).

(f) The nth excited state energy distribution is related to the distribution $\mathcal{P}_n(\Lambda)$ of the sum of the $n+1$ lengths ℓ by (68) and may be studied by the same kind of calculations, starting from

$$W_n(E) = L\rho(E) \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}q}{2\mathrm{i}\pi} \frac{\mathrm{e}^{q\,L\,N(E)}}{\cosh^{2(n+1)} \sqrt{q}} \tag{132}$$

where we have used the fact in (103) that $B = \langle \ell \rangle = N(E)^{-1} = \frac{1}{2g}\ln^2(g^2/E)$. For example, we find that the first excited state energy distribution, $W_1(E) =$

$L\rho(E)\ \varpi_1(L\,N(E))$, involves the scaling function:

$$\varpi_1(X) = \theta(X)\sum_{m=0}^{\infty}\left[\frac{\pi^4}{6}(2m+1)^4X^3 - 2\pi^2(2m+1)^2X^2 + 2\left(\frac{\pi^2}{3}(2m+1)^2+1\right)X - \frac{4}{3}\right]\mathrm{e}^{-\frac{\pi^2}{4}(2m+1)^2X} \tag{133}$$

$$= \frac{8\ \theta(X)}{3\sqrt{\pi}\,X^{3/2}}\sum_{m=1}^{\infty}(-1)^m m^2(m^2-1)\mathrm{e}^{-m^2/X} \tag{134}$$

where X is given by (122). Its limiting behaviours are

$$\varpi_1(X) \underset{X\to 0}{\simeq} \frac{32\ \theta(X)}{\sqrt{\pi}\,X^{3/2}}\mathrm{e}^{-4/X} \tag{135}$$

$$\underset{X\to\infty}{\simeq} \frac{\pi^4}{6}X^3\mathrm{e}^{-\frac{\pi^2}{4}X}. \tag{136}$$

Note, however, that for large n, since the moments of ℓ are finite, we expect a Gaussian distribution for $\mathcal{P}_n(\Lambda)$ due to the central limit theorem.

(g) The distribution $W_0(E)$ is not the distribution of the extreme value of a set of independent random variables (see appendix B) as is the case for model (A). We will return to this point at the end.

4.2.2. High energy: $E \gg g^2$. For high energy the analysis is very similar to that done for model (A), since the distribution of lengths ℓs is the same. Only the behaviours of several quantities change. The distribution $W_n(E)$ is given by (70) with $s(k) = (n+1)\frac{\pi g}{2kL^2}$. The condition to be at high energy leads to the same expression (71) with the only difference being that the localization length for the supersymmetric model at high energy reaches a constant value: $\lambda \simeq \frac{2}{g}$. Then (72) still holds with the mean value of the energy still being the free result (73) and the fluctuation being

$$\delta E_n = (n+1)\sqrt{\frac{2g\pi}{L^3}}. \tag{137}$$

The fluctuations now depend on n, compared with model (A), and its L dependence also changes. However, they remain small compared with the mean value $\langle E_n\rangle$ due to condition (71). It is also interesting to note that the ratio of the fluctuation and the mean level spacing has the same form:

$$\frac{\delta E_n^2}{\langle\Delta_n\rangle^2} \simeq \frac{2}{\pi^2}\frac{L}{\lambda} \tag{138}$$

which means that the absence of level repulsion may only be expected for the localized regime.

4.3. The case $\mu \neq 0$ at low energy ($E \ll g^2$)

We consider in this last section the case where the function $\phi(x)$ is distributed according to (77) with a non-zero mean: $\langle\phi(x)\rangle = \mu g$ (note that μ is a dimensionless parameter). We will only discuss the low-energy case since we are not expecting any modification at high energy. We have to come back for a while to equation (83) for the phase from which we want to find an equation for an additive process. Performing the change of variable $\zeta = \frac{1}{2}\ln|\tan\vartheta|$, we have, in fact, to distinguish two cases: if $(\vartheta \mod \pi) \in [0, \pi/2]$ then $1/\sin 2\vartheta = +\cosh 2\zeta$

and we get (85), but if $(\vartheta \mod \pi) \in [\pi/2, \pi]$ then $1/\sin 2\vartheta = -\cosh 2\zeta$ and we arrive at $\mathrm{d}_x\zeta = -k\cosh 2\zeta + \phi(x)$. The change of the sign in the potential is related to the fact that ζ decreases if ϑ increases if $(\vartheta \mod \pi) \in [\pi/2, \pi]$. So a more convenient change of variable is in fact:

$$\zeta = \frac{1}{2}\ln|\tan\vartheta| \qquad \text{if} \quad (\vartheta \mod \pi) \in [0, \pi/2] \tag{139}$$

$$\zeta = -\frac{1}{2}\ln|\tan\vartheta| \qquad \text{if} \quad (\vartheta \mod \pi) \in [\pi/2, \pi]. \tag{140}$$

With this convention, ζ is always travelling from $-\infty$ to $+\infty$. Then ζ obeys

$$\frac{\mathrm{d}}{\mathrm{d}x}\zeta = k\cosh 2\zeta + \phi(x) \qquad \text{if} \quad (\vartheta \mod \pi) \in [0, \pi/2] \tag{141}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\zeta = k\cosh 2\zeta - \phi(x) \qquad \text{if} \quad (\vartheta \mod \pi) \in [\pi/2, \pi]. \tag{142}$$

It follows that we have to consider alternatively two different equations. We did not mention this point before because if $\phi(x)$ is a white noise of zero mean, since ζ decorrelates when it reaches $+\infty$, the difference in the sign can always be absorbed in the white noise. We rewrite these two equations in terms of a normalized white noise $\eta(x)$ of zero mean: $\langle\eta(x)\rangle = 0$ and $\langle\eta(x)\,\eta(x')\rangle = \delta(x - x')$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\zeta = k\cosh 2\zeta + \mu g + \sqrt{g}\,\eta(x) \qquad \text{if} \quad (\vartheta \mod \pi) \in [0, \pi/2] \tag{143}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\zeta = k\cosh 2\zeta - \mu g + \sqrt{g}\,\eta(x) \qquad \text{if} \quad (\vartheta \mod \pi) \in [\pi/2, \pi]. \tag{144}$$

We have to introduce two potentials $U_1(\zeta)$ and $U_2(\zeta)$:

$$U_1(\zeta) = -\mu g\zeta - \frac{k}{2}\sinh 2\zeta \tag{145}$$

$$U_2(\zeta) = +\mu g\zeta - \frac{k}{2}\sinh 2\zeta \tag{146}$$

plotted in figure 8.

We must also introduce two variables Λ_1 and Λ_2 that give the 'times' the variable ζ needs to go from $-\infty$ to $+\infty$ in potential $U_1(\zeta)$ and $U_2(\zeta)$, respectively. For a drift $\mu = 0$ those

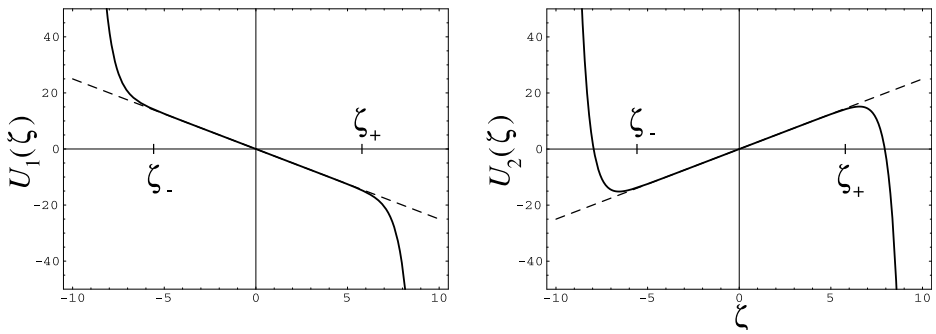

Figure 8. The two potentials in which the variable ζ evolves alternatively.

two variables have the same statistical properties as was implicit in the previous section but if $\mu \neq 0$, their distributions are different. We are now going to study these two distributions, denoted by $p_1(\Lambda_1)$ and $p_2(\Lambda_2)$. We introduce the two characteristic functions for the 'times' $\tilde{\Lambda}_{1,2}$ needed by the random process $\zeta(x)$ obeying (143) and (144) to reach $+\infty$ starting from ζ:

$$h_{1,2}(\alpha, \zeta) = \langle \mathrm{e}^{-\alpha \tilde{\Lambda}_{1,2}} \mid \zeta(0) = \zeta;\ \zeta(\tilde{\Lambda}_{1,2}) = +\infty \rangle. \tag{147}$$

We recall that

$$h_{1,2}(\alpha, -\infty) = \int_0^\infty \mathrm{d}\Lambda\ p_{1,2}(\Lambda)\, \mathrm{e}^{-\alpha\Lambda}.$$

These two functions obey (see appendix A):

$$\left((k \cosh 2\zeta \pm \mu g)\ \partial_\zeta + \frac{g}{2}\partial_\zeta^2 \right) h_{1,2}(\alpha, \zeta) = \alpha\ h_{1,2}(\alpha, \zeta) \tag{148}$$

where the upper sign (here +) corresponds to 1 and the lower sign (here −) corresponds to 2. The boundary conditions are as usual $\partial_\zeta h_{1,2}(\alpha, \zeta)\big|_{-\infty} = 0$ and $h_{1,2}(\alpha, +\infty) = 1$. Since we are dealing with the limit $E \to 0$, we can make the same approximation as for the $\mu = 0$ case, i.e. replace the previous equations by an equation for the free diffusion on the interval $[\zeta_-, \zeta_+]$, since the time spent out of this interval is negligible. Then we have to solve

$$\left(\pm\mu g\ \partial_\zeta + \frac{g}{2}\partial_\zeta^2 \right) h_{1,2}(\alpha, \zeta) = \alpha\ h_{1,2}(\alpha, \zeta) \tag{149}$$

with the boundary conditions:

$$\partial_\zeta h_{1,2}(\alpha, \zeta_-) = 0 \tag{150}$$

$$h_{1,2}(\alpha, \zeta_+) = 1. \tag{151}$$

The solution is easily found:

$$h_{1,2}(\alpha, \zeta) = \mathrm{e}^{\mp\mu(\zeta - \zeta_+)} \frac{\cosh\gamma(\zeta - \zeta_-) \pm \frac{\mu}{\gamma}\sinh\gamma(\zeta - \zeta_-)}{\cosh\gamma(\zeta_+ - \zeta_-) \pm \frac{\mu}{\gamma}\sinh\gamma(\zeta_+ - \zeta_-)} \tag{152}$$

where

$$\gamma = \sqrt{\frac{2\alpha}{g} + \mu^2}. \tag{153}$$

The characteristic functions for $\Lambda_{1,2}$ are

$$\langle \mathrm{e}^{-\alpha\Lambda_{1,2}} \rangle = h_{1,2}(\alpha, \zeta_-) = \frac{\mathrm{e}^{\pm\mu(\zeta_+ - \zeta_-)}}{\cosh\gamma(\zeta_+ - \zeta_-) \pm \frac{\mu}{\gamma}\sinh\gamma(\zeta_+ - \zeta_-)}. \tag{154}$$

We can more conveniently consider the generating function $w_{1,2}(\alpha, \zeta_-) = \ln h_{1,2}(\alpha, \zeta_-)$ for the cumulants of $\Lambda_{1,2}$. In the limit of low energy, when the condition

$$\mu(\zeta_+ - \zeta_-) = \mu \ln g/k \gg 1 \tag{155}$$

is fulfilled, a careful analysis shows that

$$w_1(\alpha, \zeta_-) = -\alpha\frac{\zeta_+ - \zeta_-}{\mu g} + \frac{\alpha^2}{2!}\frac{\zeta_+ - \zeta_-}{\mu^3 g^2} + \mathrm{O}\left(\alpha^3 \frac{\zeta_+ - \zeta_-}{\mu^5 g^3}\right) \tag{156}$$

and

$$w_2(\alpha,\zeta_-) = -\ln\left(1+\frac{\alpha}{2\mu^2 g}\mathrm{e}^{2\mu(\zeta_+-\zeta_-)}\right)+\sum_{m=1}^{\infty}\kappa_m\alpha^m\frac{(\zeta_+-\zeta_-)^m}{g^m\mu^{2m}}. \tag{157}$$

The characteristic function (156) corresponds to a sharp Gaussian distribution $p_1(\Lambda_1)$ for

$$\langle\Lambda_1\rangle = \frac{\ln(g/k)}{\mu g} \tag{158}$$

$$\langle\langle\Lambda_1^2\rangle\rangle = \frac{\ln(g/k)}{\mu^3 g^2}. \tag{159}$$

This result is not surprising since the potential U_1 is monotonic and linear in the interesting interval (see figure 8), we expect the 'time' to go through the interval to be $\frac{\text{distance}}{\text{speed}} = \frac{\zeta_+-\zeta_-}{\mu g}$ and to fluctuate weakly for large enough drift. The precise criterion for the validity of this result is that the time characterizing the motion due to the drift, $\tau_{\text{drift}} = \frac{\text{distance}}{\text{speed}}$, is much shorter than the time characterizing the motion due to the diffusion, $\tau_{\text{diff}} = \frac{(\text{distance})^2}{\text{diffusion}}$; that is $\frac{(\zeta_+-\zeta_-)}{\mu g} \ll \frac{(\zeta_+-\zeta_-)^2}{g}$, which leads to (155).

In (157), the first term gives contributions to the cumulants exponentially large in $(\zeta_+-\zeta_-)$ whereas the sum gives contributions powers of $(\zeta_+-\zeta_-)$. Then we can forget the second term and the characteristic function is the logarithm of the Laplace transform of a Poisson law $p_2(\Lambda_2) = \frac{1}{\langle\Lambda_2\rangle}\exp\bigl[-\Lambda_2/\langle\Lambda_2\rangle\bigr]$ with

$$\langle\Lambda_2\rangle \simeq \frac{1}{2\mu^2 g}\mathrm{e}^{2\mu(\zeta_+-\zeta_-)} = \frac{1}{2\mu^2 g}\left(\frac{g}{k}\right)^{2\mu}. \tag{160}$$

The fact that the distribution $p_2(\Lambda_2)$ is a Poisson law was expected from the shape of the potential $U_2(\zeta)$ (see figure 8): the potential possesses a well that is able to trap the variable ζ for a long time. We can now compare the result of the approximation we have made for $\langle\ell\rangle = \langle\Lambda_1+\Lambda_2\rangle$ with the exact result (78). We have found

$$\langle\ell\rangle \simeq \frac{\ln(g/k)}{\mu g}+\frac{1}{2\mu^2 g}\left(\frac{g}{k}\right)^{2\mu} \simeq \frac{1}{2\mu^2 g}\left(\frac{g^2}{E}\right)^{\mu}$$

whereas (78) gives

$$N(E)^{-1} \simeq \frac{1}{2g}\frac{\pi^2}{\sin^2\pi\mu}\left(\frac{4g^2}{E}\right)^{\mu}.$$

Despite the pre-factors being different, our approximation is indeed able to give the well known power-law behaviour of the integrated density of states [22, 25]: $N(E)\sim E^{\mu}$. We may also have used the Arrhenius formula (A22) to find

$$\langle\Lambda_2\rangle \simeq \frac{\pi}{\mu g}\left(\frac{2\mu g}{k}\right)^{2\mu}$$

which presents still a different pre-factor (equation (A22) does not give the correct pre-factor maybe because the potential $U_2(\zeta)$ is not smooth enough in the neighbourhood of its local minimum in the limit $k\to 0$ as is assumed to derive the Arrhenius law (A22)). However, we may distinguish two levels of approximation in what we have done.

(a) We have shown that the distribution of Λ_2 is Poisson, that could be demonstrated in a more general way following the proof presented in appendix A for the case of a potential possessing a local minimum.

(b) We have given an approximative expression of the average time $\langle\Lambda_2\rangle$ that only gives the correct behaviour with E but not the correct pre-factor.

However, we can avoid the not absolutely satisfactory approximation (b) because we know that $\langle\Lambda_1+\Lambda_2\rangle = N(E)^{-1}$ is exact. Then we may give the distribution of the length ℓ between nodes of the wavefunction:

$$P(\ell) = \int \mathrm{d}\Lambda_2\, p_2(\Lambda_2)\, p_1(\ell-\Lambda_2) \tag{161}$$

we have shown that $p_2(\Lambda_2)$ varies on a characteristic scale $\langle\Lambda_2\rangle \gg \langle\Lambda_1\rangle \gg \sqrt{\langle\langle\Lambda_1^2\rangle\rangle}$ compared with which $p_1(\Lambda_1)$ is very narrow. It follows that $P(\ell) \simeq p_2(\ell)$ apart from the behaviour at the origin we are not interested in, since it is associated with small samples. Then we conclude that

$$P(\ell) = N(E)\,\mathrm{e}^{-\ell N(E)}. \tag{162}$$

We can now give the distribution of the nth excited state which has the same form as that found for model (A) (49). Then we have:

$$W_n(E) = L\mu a_\mu E^{\mu-1}\frac{(La_\mu E^\mu)^n}{n!}\mathrm{e}^{-La_\mu E^\mu} \tag{163}$$

where the coefficient a_μ is defined by $N(E) \simeq a_\mu E^\mu$. We recall that this result is valid if (155) is fulfilled, that is $\mu\ln(g/k) \gg 1$, in addition to the fact that $k \ll g$.

It follows that the distribution involves the scaling function:

$$W_n(E) = \frac{1}{\delta\varepsilon}\,\omega_n\left(\frac{E}{\delta\varepsilon}\right) \tag{164}$$

where

$$\omega_n(X) = \mu X^{\mu-1}\frac{X^{n\mu}}{n!}\mathrm{e}^{-X^\mu} \tag{165}$$

the energy scale being

$$\delta\varepsilon = \frac{1}{(a_\mu L)^{1/\mu}}. \tag{166}$$

We may easily compute the moments:

$$\left\langle E_n^m\right\rangle = \delta\varepsilon^m\frac{\Gamma(n+1+m/\mu)}{\Gamma(n+1)}. \tag{167}$$

It is also interesting to compare the mean level spacing $\langle\Delta_n\rangle = \langle E_{n+1}-E_n\rangle$ and the fluctuations $\delta E_n = \sqrt{\langle\langle E_n^2\rangle\rangle}$. We obtain

$$\langle\Delta_n\rangle = \delta\varepsilon\frac{\Gamma(1/\mu+n+1)}{\mu\,(n+1)!}. \tag{168}$$

Then

$$\frac{\delta E_n}{\langle\Delta_n\rangle} = \mu\,(n+1)\sqrt{n!\frac{\Gamma(2/\mu+n+1)}{\Gamma(1/\mu+n+1)^2}-1} \tag{169}$$

which becomes large at small μ ($\ll 1/n^2$):

$$\frac{\delta E_n}{\langle\Delta_n\rangle} \propto \mu^{n/2+5/4}2^{1/\mu}. \tag{170}$$

At large μ the ratio reaches a constant value

$$\lim_{\mu\to\infty} \frac{\delta E_n}{\langle \Delta_n \rangle} = (n+1)\sqrt{\psi'(n+1)} \tag{171}$$

where $\psi(z)$ is the digamma function (in particular, $\psi'(1) = \pi^2/6$). For small n the fluctuations and the mean level spacing are of the same order.

The distribution (163) has the form of the distribution of the $(n+1)$th lowest variable among $N \to \infty$ statistically independent variables distributed by a law behaving like $p(E) \sim E^{\mu-1}$ (see appendix B), the same behaviour as the density of states. This remark shows that the energies behave like statistically independent ordered variables, in agreement with the expected absence of level repulsion in the localized regime.

5. Conclusion

We have considered two one-dimensional disordered models: one with diagonal disorder (A) and another with off-diagonal disorder (B).

For model (A) we have derived the distribution of the distance between consecutive nodes of the wavefunction in the limit $|E| \gg \sigma^{2/3}$, both in the negative part of the spectrum and in the positive part. Using these results we were able to find the distribution of the nth excited state (49) and (72). For $E < 0$ we have shown that (49) is a scaling law (61), similar to the extreme statistics of independent variables. If $E < 0$ the typical value of the nth energy behaves with the size of the system like $E_n^{\rm typ} \sim -\ln^{2/3} L$. The width of its distribution behaves like $\delta E_n \sim \ln^{-1/3} L$. If $E > 0$ we have found $E_n^{\rm typ} \sim 1/L^2$ and $\delta E_n \sim 1/\sqrt{L}$. Note, however, that the relative fluctuations of E_n in this latter case are small despite the fact that the behaviour with L suggests the opposite.

For model (B) we have first considered the case $\mu = 0$ (the mean value of the function entering the supersymmetric potential). The high-energy limit (universal regime) gives results similar to those for model (A). In the low-energy limit we have found the distribution for the ground state (123) and (110). We have shown that this distribution is broad, its positive moments being all dominated by the tail of the distribution: $E_0^{\rm typ} \sim {\rm e}^{-L} \ll E_0^{\rm med} \sim {\rm e}^{-\sqrt{L}} \ll \langle E_0 \rangle \sim {\rm e}^{-L^{1/3}}$. The moments have an atypical n dependence: $\langle E_0^n \rangle \sim b^{n^{2/3}}$. We have also given explicitly the distribution for the second energy level (133) and (134) and an integral representation for the other energy levels (132). For $\mu = 0$ these distributions do not have the form of the distribution of extremes of independent variables. For $\mu \neq 0$ we were able to derive the distributions $W_n(E)$ in the low-energy limit (in the high-energy limit, universal regime, we do not expect any difference with the picture obtained for $\mu = 0$). We have shown that $W_n(E)$ is a scaling function $\omega_n(E/\delta\varepsilon)$ where the energy scale behaves like $\delta\varepsilon \propto L^{-1/\mu}$. For $\mu \neq 0$ the distribution exhibits extreme value statistics as for model (A).

We now discuss the relation between the distributions $W_n(E)$ we have found and the extreme value statistics (see appendix B). We have seen that for model (A) and for model (B) with $\mu \neq 0$, $W_n(E)$ has the form of a distribution of extreme values. If we suppose that energies are behaving like statistically independent random variables, the distribution of one of them should be proportional to the density of states; the number $\mathcal{N}$ of these variables is proportional to the size of the system. If we replace $\mathcal{N}$ by $L \times \Omega$ and $p(x)$ by $\rho(E)/\Omega$ in (B1), where Ω is the total number of states per unit length (in principle, infinite since the spectrum is unbounded from above), it is easy to see that we get equation (49) obtained both for model (A) and for model (B) with $\mu \neq 0$. We now consider more specifically the case of the ground state ($n = 0$), since $W_0(E)$ is directly related to the distribution $P(\ell)$, and proceed in the opposite way we have followed until now. If we admit that the distribution of the

lowest energy is $W_0(E) = L\rho(E)\exp[-L\,N(E)]$ (see equation (49)), as a consequence of the statistical independence of the energies, then we conclude that $P(\ell)$ is Poissonian. Conversely, if $P(\ell)$ is not Poisson in the range of energy where E_0 is expected to be found, there might exist some correlations between the energies. This is indeed the case for the high-energy limit: the distribution (72) for $n = 0$, a consequence of the narrow Gaussian distribution $P(\ell)$, is valid only in the delocalized regime due to condition (71), and in this regime level repulsion occurs, as explained, due to (75). This argument can only be used for the ground state energy distribution since $W_n(E)$ is not directly related to $P(\ell)$ in this case. So the fact that (72) for $n \gg 1$ comes from a Gaussian narrow distribution $P(\ell)$ in this range of energy does not necessarily mean level repulsion. Since the distribution of the ground state for the supersymmetric model with $\mu = 0$ is $W_0(E) = L\rho(E)\varpi_0(L\,N(E))$ with $\varpi_0(X) \neq \mathrm{e}^{-X}$, this suggests that level repulsion might occur at the bottom of the spectrum. This seems to contradict the expected absence of level repulsion in localized regime since $W_0(E)$ shows that the most probable energy $E \gg E^{\mathrm{typ}} \simeq g^2\exp(-gL)$, i.e. $\lambda \simeq \frac{1}{g}\ln(g^2/E) \ll L$, are associated with localized states. However, $\lambda \simeq \frac{1}{g}\ln(g^2/E)$ is the localization length for the infinite size system, given by the inverse of the mean Lyapunov exponent $\langle\gamma\rangle = 1/\lambda$ which is going to zero if $E \to 0$. For a finite size system the Lyapunov exponent has some Gaussian fluctuations which cause fluctuations of the localization length: $\langle\langle\gamma^2\rangle\rangle \propto 1/L$. For a given L the fluctuations of the Lyapunov exponent become of the order of its mean value when the energy becomes smaller than an energy $g^2\exp(-c\sqrt{gL})$ of the order of the median value of the distribution $W_0(E)$; in this case, the large fluctuations of the Lyapunov exponent may cause delocalization and it might not be surprising that level correlation appears in that regime, which is the reason that $W_0(E)$ does not behave like the extreme value distribution of independent variables as is the case for $\mu \neq 0$ or for model (A). Nevertheless, a deeper understanding of the level correlations seems to be needed for the supersymmetric case with $\mu = 0$.

Since model (B) is related to the problem of classical diffusion in a random medium, it would be interesting to know whether these results have an application in this case. The energies of the Hamiltonian should be related to relaxation times [23]. The spectral properties near $E = 0$ are important for the disordered spin chain models [21], the distribution $W_0(E)$ of the lowest mode of the Hamiltonian (76) might also have some interest in that context.

Acknowledgments

I am grateful to Alain Comtet for having motivated my interest in this problem and with whom I had very interesting discussions (in particular, he brought to my attention [37, 41, 46]). I thank Gleb Oshanin for interesting remarks. I also had stimulating discussions with Marc Bocquet, Jean-Marc Luck and Stéphane Ouvry. This work was supported by the Swiss National Science Foundation and by the TMR Network Dynamics of Nanostructures.

Appendix A. Distribution of the escape time for a diffusive particle trapped in a well

In this appendix we recall known results about the trapping of a Brownian particle by a well and the distribution of the escape time. In the regime of interest here the average escape time is given by the Arrhenius law. Most of what is in this appendix may be found in standard textbooks like [47] and we summarize here some ideas needed throughout this paper.

We consider a particle whose position $x(t)$ obeys the following stochastic differential equation:

$$\mathrm{d}x(t) = A(x)\,\mathrm{d}t + \sqrt{B(x)}\,\mathrm{d}W(t) \qquad \text{(Ito)} \tag{A1}$$

where $W(t)$ is a normalized Wiener process: $\langle W(t)\rangle = 0$ and $\langle W(t)W(t')\rangle = \min(t,t')$. This equation is understood in the Ito sense. The propagator $p(x',t|x,0)$ for the diffusion (conditional probability for the particle to be at x' at time t, starting from x at initial time 0) obeys the backward Fokker–Planck equation (BFPE):

$$\partial_t p(x',t|x,0) = G_x\, p(x',t|x,0) \tag{A2}$$

where

$$G_x = A(x)\partial_x + \frac{1}{2}B(x)\partial_x^2 \tag{A3}$$

is the BFPE generator.

Let us consider a process starting from a point $x(0) = x$ belonging to the interval $[a,b]$. Let us call T the time needed by the particle to leave this interval. The probability that the particle is still in the interval after time T is

$$\int_a^b \mathrm{d}x'\, p(x',T|x,0) = \int_T^\infty \mathrm{d}T'\, P_x(T') \tag{A4}$$

where $P_x(T)$ is the probability density for the escape time. Let us define $T_n(x) = \langle T^n \mid x(0) = x;\ x(T) = a \text{ or } b\rangle$, the nth moment of the escape time from the interval $[a,b]$. It follows from (A4) that

$$G_x\, T_n(x) = -nT_{n-1}(x). \tag{A5}$$

Since the particle may escape from both sides of the interval, we have to impose the following boundary conditions: $T_n(a) = T_n(b) = 0$. This absorbing boundary conditions mean that the particle leaves the interval and never comes back as soon as it reaches one of the two edges. In this sense we are interested in a time of first exit, relevant for the question considered in this paper.

In the following we will be interested in the more simple case of a reflection condition at one side of the interval, $x = a$, so that the particle may only escape from the side $x = b$:

$$T_n(x) = \langle T^n \mid x(0) = x;\ x(T) = b\rangle. \tag{A6}$$

The reflecting boundary condition for the BFPE is $\partial_x p(x',t|x,0)|_{x=a} = 0$ [47] which implies the following boundary conditions for the moments:

$$\partial_x T_n(a) = 0 \tag{A7}$$

$$T_n(b) = 0. \tag{A8}$$

In this case it is easy to construct from (A5) the moments:

$$T_n(x) = 2n\int_x^b \mathrm{d}x'\, \frac{1}{\psi_0(x')}\int_a^{x'} \mathrm{d}x''\, \frac{\psi_0(x'')}{B(x'')}\, T_{n-1}(x'') \tag{A9}$$

where

$$\psi_0(x) = \exp\int^x \mathrm{d}x'\, \frac{2A(x')}{B(x')}. \tag{A10}$$

Keeping in mind that $T_0(x) = 1$, it follows that the moments may be computed recursively.

We also introduce the generating function for the distribution $P_x(T)$:

$$h(\alpha,x) = \langle \mathrm{e}^{-\alpha T} \mid x(0) = x,\ x(T) = b\rangle \tag{A11}$$

which may be used to analyse the distribution. It is clear from (A5) that it obeys the following differential equation:

$$G_x\, h(\alpha, x) = \alpha\, h(\alpha, x). \tag{A12}$$

The boundary conditions for reflection in a and escape (absorption) at b are:

$$\partial_x h(\alpha, a) = 0 \tag{A13}$$

$$h(\alpha, b) = 1. \tag{A14}$$

Instead of considering (A12)–(A14) it can be more convenient to write an integral equation for h:

$$h(\alpha, x) = 1 - 2\alpha \int_x^b \mathrm{d}x' \, \frac{1}{\psi_0(x')} \int_a^{x'} \mathrm{d}x'' \, \frac{\psi_0(x'')}{B(x'')}\, h(\alpha, x''). \tag{A15}$$

We now derive the distribution $P_x(T)$ when the particle is trapped by the well of a potential $U(x)$, in the small noise limit. We consider the simpler situation of a stochastic differential equation of the form

$$\mathrm{d}x(t) = -U'(x)\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}W(t) \tag{A16}$$

where D is the diffusion constant. The shape of the potential of interest is depicted in figure A1. The potential $U(x)$ has a local minimum x_0 from the neighbourhood of which the particle starts. Let us call x_i the initial position of the particle. The potential has at $x_1 > x_0$ a local maximum. We are interested in the time that the particle needs to escape the well, jumping over the potential barrier due to a fluctuation. The interval $[a, b]$ to be considered has to include the well and the barrier. As will be clear in the following we have to consider a limit b such that the distance $b - x_1$ is sufficiently large (the relevant length scale is given by the curvature of $U(x)$ at x_1). For similar reasons the distance $x_0 - a$ has to be sufficiently large as well. Apart from this two restrictions the precise positions of a and b are of little importance.

Let us analyse the behaviour of the moments when the diffusion D is small compared with the height of the barrier. Equation (A9) now takes the form:

$$T_n(x_i) = \frac{2n}{D} \int_{x_i}^b \mathrm{d}x\, \mathrm{e}^{2U(x)/D} \int_a^x \mathrm{d}x'\, \mathrm{e}^{-2U(x')/D}\, T_{n-1}(x'). \tag{A17}$$

From the shape of the potential of figure A1, it is clear that the integral over x is dominated by the neighbourhood of x_1 and the integral over x' by the neighbourhood of x_0. Accordingly, we have

$$T_n(x_i) \simeq \frac{2n}{D} T_{n-1}(x_0) \int_{x_i}^b \mathrm{d}x\, \mathrm{e}^{2U(x)/D} \int_a^{x_1} \mathrm{d}x'\, \mathrm{e}^{-2U(x')/D}. \tag{A18}$$

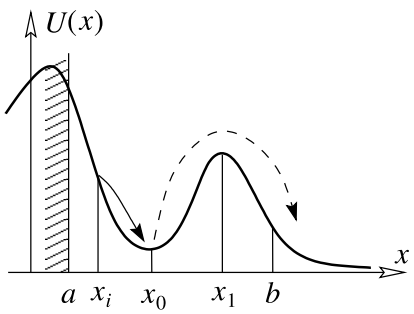

Figure A1. The particle starts from x_i and escapes the well after a time T.

Since we are dealing with the limit $D \to 0$, the two integrals may be estimated by the steepest-descent method. Expanding the potential in the neighbourhood of its two extrema:

$$U(x) \underset{x\sim x_0}{\simeq} U(x_0) + \frac{(x-x_0)^2}{2\delta_0^2} \tag{A19}$$

$$U(x) \underset{x\sim x_1}{\simeq} U(x_1) - \frac{(x-x_1)^2}{2\delta_1^2} \tag{A20}$$

we eventually find

$$T_n(x_i) \simeq n\, T_{n-1}(x_0)\, 2\pi\, \delta_0\, \delta_1\, \mathrm{e}^{2\frac{U(x_1)-U(x_0)}{D}}. \tag{A21}$$

For $n = 1$, we recover the well known Arrhenius law [47]:

$$T_1(x_0) \simeq 2\pi\, \delta_0\, \delta_1\, \mathrm{e}^{2\frac{U(x_1)-U(x_0)}{D}}. \tag{A22}$$

Let us remark that for $b = x_1$ we find half of this result. It is now straightforward to see that the moments are:

$$T_n(x_0) \simeq n!\, [T_1(x_0)]^n \tag{A23}$$

i.e. the moments of a Poisson law. It follows that the escape time from a deep well is distributed by

$$P_{x_i}(T) = \frac{1}{T_1(x_0)} \mathrm{e}^{-T/T_1(x_0)}. \tag{A24}$$

Let us stress that the initial condition x_i played no role in the previous discussion: $T_n(x_i) \simeq T_n(x_0)$.

The same analysis may also be performed using (A15).

Appendix B. Extreme value statistics

We give here a brief discussion on extreme order statistics, details of which can be found in [37, 41, 46].

We consider a set of $\mathcal{N}$ statistically independent variables x_n, distributed according to the law $p(x)$. Then the $\mathcal{N}$ variables are ordered and we call $w_n(x)$ the probability density of the nth variable among these variables. We have

$$w_n(x)\,\mathrm{d}x = nC_{\mathcal{N}}^n \left(\int_{-\infty}^{x} \mathrm{d}x'\, p(x')\right)^{n-1} p(x)\,\mathrm{d}x \left(\int_{x}^{+\infty} \mathrm{d}x'\, p(x')\right)^{\mathcal{N}-n}. \tag{B1}$$

We are now interested in the behaviour of this expression in the limit $\mathcal{N} \to \infty$. Three kinds of distribution are usually distinguished.

Type I. $p(x)$ is unbounded from below and decays exponentially. In the limit $\mathcal{N} \to \infty$ the distribution $w_n(x)$ is expected to be peaked around a value $x_n^{\mathrm{typ}} \to -\infty$ so that we may use the fact that $p(x)$ may be locally approximated by an exponential in the neighbourhood of x_n^{typ}. Then we get, up to a rescaling $w_n(x)\,\mathrm{d}x = v_n(y)\,\mathrm{d}y$:

$$v_n(y) = \frac{n^n}{(n-1)!} \exp\left(ny - n\mathrm{e}^y\right) \tag{B2}$$

where y is the rescaled variable. The relation between x and the scaling variable y depends on the distribution $p(x)$ but not the scaling function $v_n(y)$.

Type II. $p(x)$ has a power-law tail. We do not discuss this case which is not relevant for what is done in this paper.

Type III. $p(x)$ is bounded from below. We assume its support starts at $x = 0$ and it behaves like $p(x) \sim x^{\mu-1}$ for small x. Then, up to a rescaling of the variable, we end with

$$v_n(y) = \frac{n^n \mu}{(n-1)!} y^{\mu n-1} \mathrm{e}^{-ny^\mu}. \tag{B3}$$

References

[1] Deych L I, Lisyansky A A and Altshuler B L 2000 Single parameter scaling in one-dimensional localization revisited *Phys. Rev. Lett.* **84** 2678

[2] Jayannavar A M, Vijayagovindan G V and Kumar N 1989 Energy dispersive backscattering of electrons from surface resonances of a disordered medium and $1/f$ noise *Z. Phys.* B **75** 77

[3] Comtet A and Texier C 1997 On the distribution of the Wigner time delay in one-dimensional disordered systems *J. Phys. A: Math. Gen.* **30** 8017

[4] Texier C and Comtet A 1999 Universality of the Wigner time delay distribution for one-dimensional random potentials *Phys. Rev. Lett.* **82** 4220

[5] Texier C 1999 Quelques aspects du transport quantique dans les systèmes désordonnés de basse dimension *PhD Thesis* Université Paris VI

[6] Ramkrishna S A and Kumar N 2000 Imaginary potential as a counter of delay time for wave reflection from a one-dimensional random potential *Phys. Rev.* B **61** 3163

[7] Titov M and Fyodorov Y V 2000 Time-delay correlations and resonances in one-dimensional disordered systems *Phys. Rev.* B **61** R2444

[8] Bolton-Heaton C J, Lambert C J, Fal'ko V I, Prigodin V and Epstein A J 1999 Distribution of time constants for tunneling through a one-dimensional disordered chain *Phys. Rev.* B **60** 10569

[9] Gogolin A O 2000 Analytic structure of one-dimensional localization theory: re-examining Mott's law *Phys. Rev. Lett.* **84** 1760

[10] Schmidt H 1957 Disordered one-dimensional crystals *Phys. Rev.* **105** 425

[11] Frisch H L and Lloyd S P 1960 Electron levels in a one-dimensional random lattice *Phys. Rev.* **120** 1175

[12] Bychkov Y A and Dykhne A M 1966 Electron spectrum in a one-dimensional system with randomly arranged scattering centers *Piz. Zh. Eksp. Teor. Fiz.* **3** 313

[13] Lifshits I M, Gredeskul S A and Pastur L A 1988 *Introduction to the Theory of Disordered Systems* (New York: Wiley)

[14] Halperin B I 1965 Green's functions for a particle in a one-dimensional random potential *Phys. Rev.* **139** A104

[15] Berezinskiĭ V L 1974 Kinetics of a quantum particle in a one-dimensional random potential *Sov. Phys.–JETP* **38** 620

[16] Vorov O K and Vagov A V 1995 Problem of a quantum particle in a random potential on a line revisited *Phys. Lett.* A **205** 301

[17] Antsygina T N, Pastur L A and Slyusarev V A 1981 Localization of states and kinetic properties of one-dimensional disordered systems, *Sov. J. Low Temp. Phys.* **7** 1–21

[18] Witten E 1981 Dynamical breaking of supersymmetry *Nucl. Phys.* B [FS] **188** 513

[19] Ziman T 1982 Localization and spectral singularities in random chains *Phys. Rev. Lett.* **49** 337

[20] Fabrizio M and Mélin R 1997 Coexistence of antiferromagnetism and dimerization in a disordered spin-Peierls model: exact results *Phys. Rev. Lett.* **78** 3382

[21] Steiner M, Fabrizio M and Gogolin A O 1998 Random mass Dirac fermions in doped spin-Peierls and spin-ladder systems: one-particle properties and boundary effects *Phys. Rev.* B **57** 8290

[22] Bouchaud J-P, Comtet A, Georges A and le Doussal P 1990 Classical diffusion of a particle in a one-dimensional random force field *Ann. Phys., NY* **201** 285–341

[23] le Doussal P, Monthus C and Fisher D S 1999 Random walkers in one-dimensional random environments: Exact renormalization group analysis *Phys. Rev.* E **59** 4795

[24] Dyson F J 1953 The dynamics of a disordered linear chain *Phys. Rev.* **92** 1331

[25] Ovchinnikov A A and Erikmann N S 1977 *Zh. Eksp. Teor. Fiz.* **73** 650

[26] Bouchaud J-P, Comtet A, Georges A and le Doussal P 1987 The relaxation-time spectrum of diffusion in a one-dimensional random medium: an exactly solvable case *Europhys. Lett.* **3** 653

[27] Comtet A, Desbois J and Monthus C 1995 Localization properties in one-dimensional disordered supersymmetric quantum mechanics *Ann. Phys., NY* **239** 312–50
[28] Monthus C 1995 Étude de quelques fonctionnelles du mouvement brownien et de certaines propriétés de la diffusion unidimensionnelle en milieu aléatoire *PhD Thesis* Université Paris VI
[29] Balents L and Fisher M P A 1997 Delocalization transition via supersymmetry in one dimension *Phys. Rev.* B **56** 12970
[30] Comtet A and Texier C 1998 One-dimensional disordered supersymmetric quantum mechanics: a brief survey *Supersymmetry and Integrable Models (Chicago, IL 1997) (Lecture Notes in Physics)* ed H Aratyn *et al* (Berlin: Springer) p 313
(Comtet A and Texier C 1997 *Preprint* cond-mat/9707313)
[31] Bocquet M 1999 Some spectral properties of the one-dimensional disordered Dirac equation *Nucl. Phys.* B [FS] **546** 621
[32] Bocquet M 2000 Chaînes de spins, fermions de Dirac et systèmes désordonnés *PhD Thesis* École Polytechnique
[33] Brouwer P W, Mudry C and Furusaki A 2000 Density of states in coupled chains with off-diagonal disorder *Phys. Rev. Lett.* **84** 2913
[34] Luck J-M 1992 *Systèmes Désordonnés Unidimensionnels* Aléa Saclay
[35] Grenkova L N, Molčanov S A and Sudarev J N 1983 On the basic states of one-dimensional disordered structures *Commun. Math. Phys.* **90** 101
[36] McKean H P 1994 A limit law for the ground state of Hill's equation *J. Stat. Phys.* **74** 1227
[37] Gumbel E J 1958 *Statistics of Extremes* (New York: Columbia University)
[38] Molčanov S A 1981 The local structure of the spectrum of the one-dimensional Schrödinger operator *Commun. Math. Phys.* **78** 429
[39] Monthus C, Oshanin G, Comtet A and Burlatsky S 1996 Sample-size dependence of the ground-state energy in a one-dimensional localization problem *Phys. Rev.* E **54** 231
[40] Jona-Lasinio G 1983 Qualitative theory of stochastic differential equations and quantum mechanics of disordered systems *Helv. Phys. Acta* **56** 61
[41] Gumbel E J 1954 Statistical theory of extreme values and some practical applications *Nat. Bur. Stand. Appl. Math. Ser. (NBS Circular no 33)* (Washington, DC: US Govt Printing Office)
[42] Bunder J E and McKenzie R H 2000 Derivation of the probability distribution function for the local density of states of a disordered quantum wire via the replica trick and supersymmetry *Preprint* cond-mat/0006291
[43] Altshuler B L and Prigodin V N 1989 Distribution of local density of states and NMR line shape in a one-dimensional disordered conductor *Sov. Phys.–JETP* **68** 198
[44] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals, Series and Products* 5th edn (New York: Academic)
[45] Oshanin G 2000 Private communication
[46] Galambos J 1978 *Asymptotic Theory of Extreme Order Statistics* (New York: Wiley)
[47] Gardiner C W 1989 *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Berlin: Springer)